Prof. Dr Satyanarayana Bhavanari
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Dimension and Graph Theoretic Aspects of Rings (Monograph)

Prime Ideal, Finite Goldie Dimension, Zero Square Dimension, Zero Square Fuzzy Ideal, Principal Ideal Graph, Prime Graph of a Ring

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CONTENTS

DIMENSION AND GRAPH THEORETIC ASPECTS OF RINGS

(Monograph)

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CONTENTS

DIMENSION AND GRAPH THEORETIC ASPECTS OF RINGS

(Monograph)

PREFACE

This monograph entitled "Dimension and Graph Theoretic Aspects of Rings", is divided into seven chapters.

In Chapter 1, we present some fundamental notions and results that are useful in the subsequent chapters. In Chapter-2, we place attention on the concepts: "sp-system" "m-system" and "g-system", and present few fundamental important results in ring theory. In Chapter-3, as the main theorem, we prove an important dimension condition on Modules with finite Goldie Dimension.
In Chapter-4, we consider the zero square rings that are studied by Stanley [1]. We define and study the concepts zero square ring of type-1/type-2. In Chapter-5, we obtain some interesting results related to the concept 'finite dimension' in the theory of associative rings R with respect to two sided ideals. In Chapter-6, we define a new type of graph (called 'Principal Ideal Graph', denoted by PIG(R)) related to a given associative ring R. We discuss few fundamental important relations between rings and graphs with respect to the properties: simple ring, complete graph, etc. In Chapter-7, we define a new concept 'Prime Graph of R' (denoted by PG(R)). We present some examples. We discuss some fundamental important results related to PG(R).

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The first author place on record his deep sense of gratitude to his parents: Bhavanari Ramakotaiah (a teacher in an elementary school at the village named Madugula) (Father), and Bhavanari Anasuryamma (house hold) (Mother), without whose constant encouragement and help it would not have been possible for him to pursue higher studies in Mathematics. Also he thank his wife: Bhavanari Jaya Lakshmi, and his children: Mallikharjun B.Tech., Satyasri (IV Year MBBS, China), and Satya Gnyana Sri (Student, 10+) for their constant patience with him and helping in bringing out better output.

The second author expresses his deep sense of gratitude and appreciation to his parents and other family members for their inspiration and without whose constant encouragement, it would not have been possible for him to pursue higher studies in Mathematics.

Prof. Dr Satyanarayana Bhavanari
and
Dr Nagaraju Dasari

INTRODUCTION

In recent decades interest has arisen in algebraic systems with binary operations addition and multiplication. 'Ring' is one of such system. A ring is an algebraic system (R, +, .) satisfying the conditions:

i) (R, +) is an Abelian group;
ii) (R, .) is a semi-group; and
iii) $a(b + c) = ab + ac$, and $(a + b)c = ac + bc$ for all $a, b, c \in R$.

Ring theory became an important part of Algebra.

Modern Algebra presently, the basis for developing several new areas mentioned below. The past 30 years have seen an enormous expansion in several new areas of technology. These new areas include Digital Computing, Data Communication, Sequential Machines, Computer Systems and Radar Solar Systems. Work in each of these areas relies heavily on Modern Algebra. This fact has made the study of Modern Algebra important to Applied Mathematicians, Engineers and Scientists who use Digital Computers or who work in the other areas of Technology mentioned above.

Throughout this monograph R stands for a fixed associative (not necessarily commutative) ring.

The concepts: prime ideal and prime radical play a vital role in the theory of rings. While studying the prime ideals, semiprime ideals, prime radical, the authors introduced the concepts: m-systems, sp-systems and used in proving the related results. Much concentration was not placed by the earlier authors on the study of the concepts: m-system & g-system. We study these concepts further in a part of this monograph and present several new results.

It is well known that the dimension of a vector space is defined as the number of elements in its basis. One can define a basis of a vector space as a maximal set of linearly independent vectors. This concept was generalized to modules over rings by Goldie, later this generalized concept to modules is called as Goldie dimension. Goldie [1] introduced the concept of Finite Goldie Dimension (FGD, in short) in modules over associative rings which is a generalization of the dimension of vector space. Goldie proved a structure theorem for modules which states that "a module with FGD contains uniform submodules $U_1, U_2, \dots U_n$ whose sum is direct and essential in M". The number n obtained here is independent of the choice of $U_1, U_2, \dots U_n$ and it is called as Goldie Dimension of M. Later this dimension theory in modules over rings was studied and developed by Reddy & Satyanarayana [1]. We study further and obtain few important results related to modules with finite Goldie Dimension.

The concept of zero square ring was introduced and studied by Stanley [1]. In this monograph, we introduce the concepts zero square ring of type-1/type-2. The zero square ring of type-2 coincides the zero square ring studied by Stanley. In this monograph, we also study zero square ideals and direct products of zero square rings.

Success of fuzzy logic in a wide range application inspired much interest in fuzzy logic among Mathematicians. Lotfi. A. Zadeh (a professor in Electrical Engineering and Computer Science at University of California, Berkeley) (July 1964) introduced a theory whose objects called 'fuzzy sets' (are sets with boundaries that are not precise). In a narrow sense fuzzy logic refers to a logical system that generalizes classical two-valued logic for reasoning under uncertainty. Prof. Zadeh believed that all real world problems could be solved with more efficient and analytic methods by using the concept fuzzy sets. The fuzzy boom (1987 to present) in Japan was a result of the close collaboration and technology transfer between Universities and Industries. In 1988 the Japanese Government launched a careful feasibility study about establishing national research projects on fuzzy logic involving both Universities and Industries. As a result the Japan is able to manufacture fuzzy vacuum cleaner, fuzzy rice cookers, fuzzy refrigerators, fuzzy washing machines, and others. After the introduction of Fuzzy set by Zadeh [1], the researchers in mathematics were trying to introduce and study this concept of fuzzyness in different mathematical systems under study. Fuzzy dimension in Modules was introduced and studied by Satyanarayana, Godloza & Mohiddin [1]. In this monograph, we introduce and study the concept 'zero square fuzzy ideal'.

The existing literature (related to module theory) tells about dimension theory for ideals (that is, two sided ideals) in case of commutative rings, and left (or right) ideals in case of associative (but not commutative) rings. So at present, we can understand the structure theorem for R in terms of one sided ideals only (that is, if R has FGD with respect to left (or right) ideals, then there exists n uniform left (or right) ideals of R whose sum is direct and essential in R). This result can not say about the structure theorem for associative rings in terms of two sided ideals. On the way of trying this structure theorem for associative rings, Satyanarayana, Nagaraju, Balamurugan & Godloza [1] defined the concept of Finite Dimension of a ring with respect to two sided ideals (FDI, in short) in associative rings and obtained the structure theorem and other few related results.

Graph Theory has been discovered independently. Euler (1707 - 1782) first introduced the concept of a graph in describing the Konigsberg bridges problem. Subsequent discoveries of Graph Theory by Kirchhoff (1824 - 1887) and Cayley, also had their roots in the physical world. Kirchhoff investigations of electrical networks lead to his developments of the basic concepts and theorems concerning trees in graphs. In 1857, Cayley discovered the important class of graphs by considering the changes of variables in the differential calculus. He considered trees arising from the enumeration of the organic chemical isomers ($C_n H_{2n+2}$). Hamilton (1805 - 1865) invented the city route puzzle in 1850's.

The last three decades have witnessed an upsurge of interest and activity in Graph Theory, particularly among applied mathematicians and engineers. Clear evidence of this is to be found in an unprecedented growth in the number of papers and books being published in the field. Graph Theory has a surprising number of applications in many developing areas. It has become fashionable to mention that there are applications of Graph Theory to solve practical problems in electrical network analysis, in circuit layout in data structures, Chemistry, Communication science, Computer Technology, Electrical and Civil Engineering.

One of the beauties of Graph Theory is that it depends very little on the other branches of Mathematics. There are several reasons for the acceleration of interest in Graph Theory. One of the attractive features of Graph Theory is its inherent pictorial character.

In this monograph throughout we consider an associative ring R (not necessarily commutative).

The content of the monograph is divided into seven Chapters

Chapter-1 is a collection of fundamental notions and results that are useful in the subsequent chapters.

In Chapter-2, we place attention on the concepts: "sp-system" "m-system" and "g-system", and present few fundamental important results in ring theory. We also include some necessary examples. We prove that if f: $R \rightarrow R^1$ is a ring epimorphism, and $S \subseteq R$, then S is an sp-system (m-system, respectively) in R if and only if f(S) is an sp-system (m-system, respectively) in R^1. We include an example of an sp-system which is not an m-system. A mapping g from R to the set of all ideals of R that satisfy the conditions: (i) $a \in g(a)$; and (ii) $x \in g(a) + A \Rightarrow g(x) \subseteq g(a) + A$, for any element $a \in R$ and for any ideal A of R, is called an ideal mapping. With respect to a given ideal mapping, we define g-system in rings and proved that if f: $R \rightarrow R^1$ is a ring epimorphism and $S \subseteq R$, then S is a g-system in R if and only if f(S) is a g-system in R^1. We present an example of a g-system which is not an m-system. The intersection of all g-prime ideals of R is called as g-prime radical of R (we denote this g-prime radical by r_g). As usual, r(R) denotes the prime radical of R. We presented some relations between the concepts: m-system, g-system and ideals. We prove that if R and g satisfy the condition "every g-system is an m-system", then (i) An ideal P of R is a g-prime ideal if and only if it is a prime ideal; (ii) $r_g(R) = r(R)$ (that is, the g-prime radical of the ring R is equal to the prime radical of the ring R). Finally, in this chapter, we obtain a result that states that if A is an ideal of a ring R such that $A \supseteq g(0)$, then the following statements are true: (i) Every g-system in R/A is an m-system; (ii) An ideal of R/A is a prime ideal if and only if it is a g-prime ideal; and (iii) $r_g(R/A) = r(R/A)$. The content of Chapter-2 formed the paper entitled "Some Results on m-systems and g-systems in Rings" which was published in a journal entitled "South East Asian Bulletin of Mathematics", 34 (2010) 461-468.

In Chapter-3, as the main theorem, we prove that if M is a module with Finite Goldie Dimension (FGD) and K_1, K_2 are two submodules of M such that $K = K_1 \cap K_2$ is a complement, then $\dim K_1 + \dim K_2 = \dim(K_1 + K_2) + \dim (K_1 \cap K_2)$. This main theorem is available in the literature (ref. Camillo and Zelmanowitz [1, 2]) but the proof we present here is totally different from the very general and long proof of Camillo and Zelmanowitz [1, 2]. The content of Chapter-3 formed the paper entitled "A Theorem on Modules with Finite Goldie Dimension" which was published in Soochow Journal of Mathematics, 32 (2006) 311-315.

In Chapter-4, we consider the zero square rings that are studied by Stanley [1]. We prove several interesting results related to this concept. We define and study the concepts zero square ring of type-1/type-2. Zero square ring of type-2 is same as the zero square ring studied by the earlier authors. We present some illustrations. Every zero square ring of type-1 is a zero square ring of type-2, but the converse need not be true, in general. We define and study zero square ideal of type-1/type-2. We verify that the class of all zero square rings R of type-1 for which $R^2 \not\subseteq I$ for all non-zero ideals I of R, is homomorphically closed. We prove that the direct product of zero square rings R_i, $1 \leq i \leq k$ of type-1 is also a zero square ring of type-1, but the converse need not be true, in general. We obtain some important consequences. We introduce the concept zero square fuzzy ideal. We prove that there exists a bijection between the set of all zero square ideals of R and the set of all zero square fuzzy ideals μ of R, where the image of μ is equal to two element set consisting of two numbers α, β in the closed interval [0, 1] with $\alpha < \beta$.

The first part of Chapter-4 formed the paper entitled "Ideals and Direct Product of Zero Square Rings", which was published in the journal "Eastasian Mathematical Journal" 24 (2008) 377-387. The second part of this chapter formed the paper entitled "Fuzzy Ideals of Zero Square Rings", which was accepted for publication in "International Journal of Fuzzy Mathematics", 2011.

In Chapter-5, we obtain some interesting results related to the concept 'finite dimension' defined (in Satyanarayana, Nagaraju, Balamurugan, and Godloza [1]) in the theory of associative rings R with respect to two sided ideals. It is known that (Corollary 4.5 of Satyanarayana, Nagaraju, Balamurugan, and Godloza [1]) if a ring R has finite dimension on ideals, then there exist uniform ideals U_i, $1 \leq i \leq n$ of R such that the sum $U_1 \oplus U_2 \oplus \ldots \oplus U_n$ is essential in R. This n is independent of choice of uniform ideals and we call it as dimension of R (we write dim R, in short). In this Chapter we verify that if R and S are two rings that are isomorphic, then dim R = dim S. We obtain that if K is a complement ideal of R, and R has finite dimension on ideals (FDI, in short), then R/K has FDI. We include an example of an ideal K of R with dim R/K $\neq$ dim R – dim K. Finally, we prove that if K is a complement ideal of R, then dim R/K = dim R – dim K. We also define zero square dimension for a zero square ring R and obtained few important results.

The content of the first part of Chapter-5 formed the paper entitled "Some Dimension Conditions in Rings with Finite Dimension", which was published in "PMU Journal of Humanities and Sciences", 1 (2010) 69-75. The Second part of Chapter-5 formed the paper entitled "On the Dimension of the Quotient Ring R/K where K is a Complement", which was published in "International Journal of Contemporary Advanced Mathematics", 1 (No. 2) (2011) 16-22.

In Chapter-6, we define a new type of graph (called 'Principal Ideal Graph', denoted by PIG(R)) related to a given associative ring R. We present some examples. We discuss few fundamental important relations between rings and graphs with respect to the properties: simple ring, complete graph, etc. We also observe that if R and S are isomorphic rings, then the related principal ideal graphs are isomorphic, but the converse is not true. We define an equivalence relation on a given ring R and obtain a one-to-one correspondence between the set of all equivalence classes and the set of all connected components of PIG(R). We introduce the concept 'full Hamiltonian decomposition' for a general graph, and prove that there exists a full Hamiltonian decomposition for PIG(R).

The content of Chapter-6 formed the paper entitled "Some Results on Principal Ideal Graph of a Ring", which was accepted for its publication in the journal "African Journal of Mathematics and Computer Science Research", 2011.

In Chapter-7, we define a new concept 'Prime Graph of R' (denoted by PG(R)). We present some examples. We discuss some fundamental important results related to PG(R). We prove that if R is a semiprime ring, then R is a prime ring if and only if PG(R) is a tree. We observe several properties of PG(R) with respect to the properties like: zero divisors, nilpotent elements in R. We apply some graph theoretic concepts to integral domains.

First part of Chapter-7 formed the paper entitled "Prime Graph of a Ring", which was published in a Journal entitled "Journal of Combinatorics, Information and System Sciences", 35 (2010) 27-42 . The second part of Chapter-7 is communicated (in the form of a Research Paper) to a Journal for its publication.