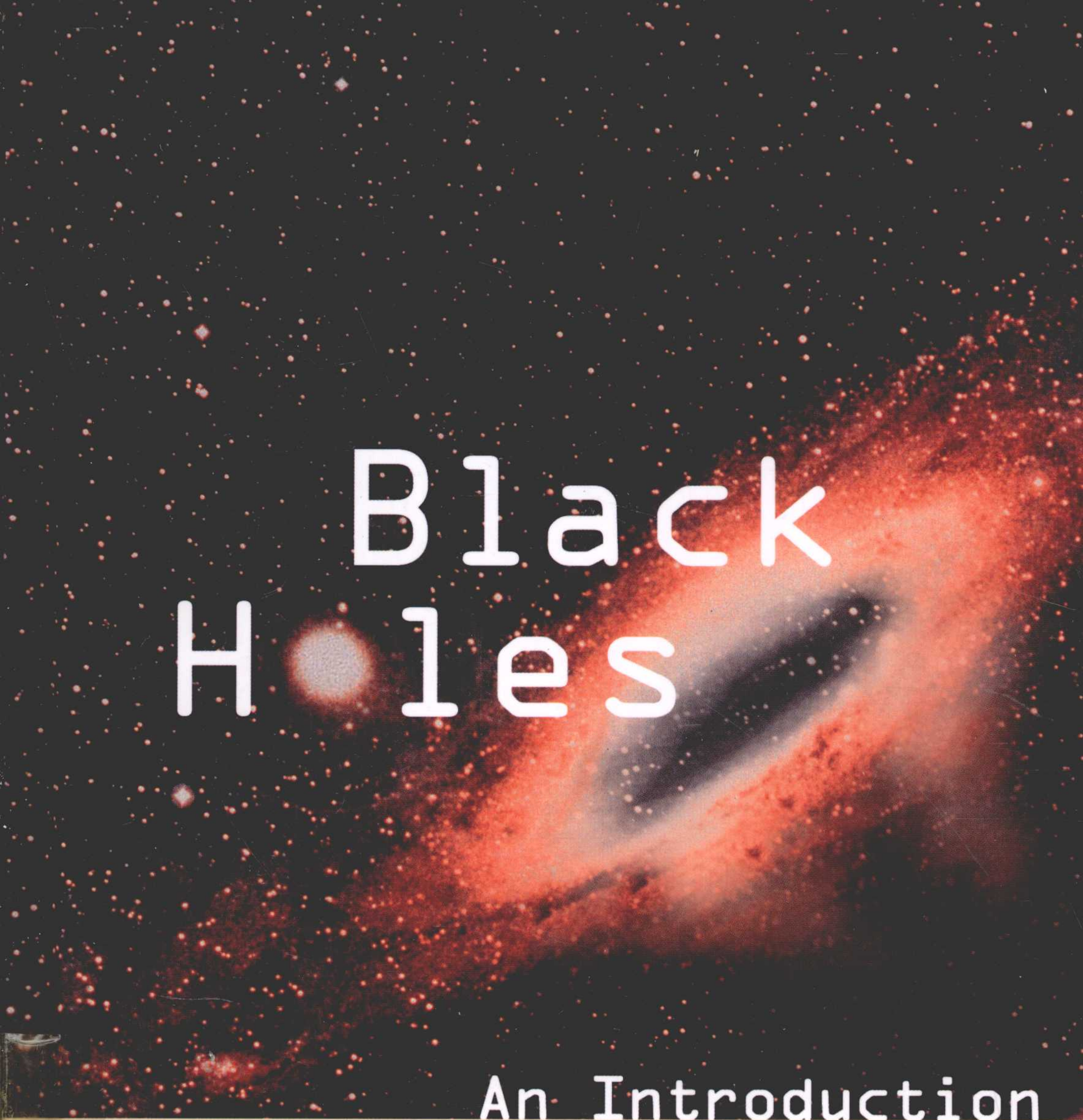


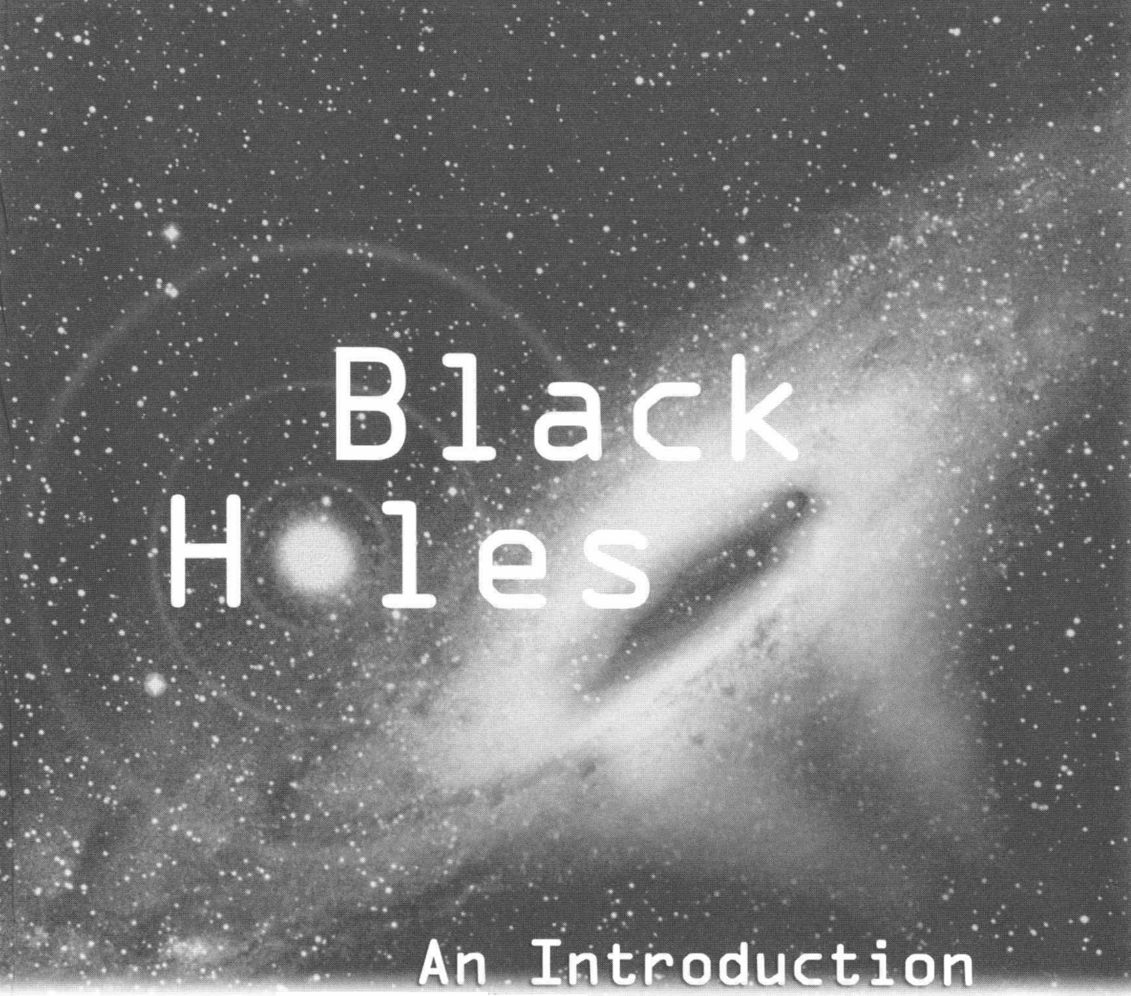
Black Holes

An Introduction

Second Edition

Derek Raine
Edwin Thomas

Imperial College Press



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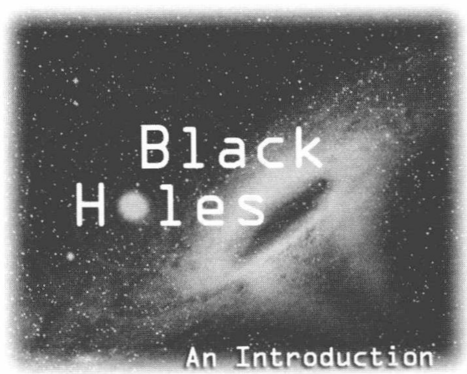

Second Edition

Preface

New science can seem quite weird at first: Newton's mystical action-at-a-distance; Maxwell's immaterial oscillations in a vacuum; the dice-playing god of quantum mechanics. In due course however we come to accept how the world is and teach it to our students. Our acceptance, and theirs, comes principally through mastery of the hard details of calculations, not from generalised philosophical debate (although there is a place for that later).

Black holes certainly seem weird. We know they (almost certainly) exist, although no-one will ever 'see' one. And they appear to play an increasingly central role both in astrophysics and in our understanding of fundamental physics. It is now almost 90 years since the Schwarzschild solution was discovered, 80 years since the first investigations of the Schwarzschild horizon, 40 years on from the first singularity theorems and perhaps time that we can begin to dispel the weirdness and pass on something of what we understand of the details to our undergraduate students. This is what we attempt in this book. We have tried to focus on the aspects of black holes that we think are generally accessible to physics undergraduates who may not (or may) intend to study the subject further. Many of the calculations in this book can be done more simply using more sophisticated tools, but we wanted to avoid the investment of effort from those for whom these tools would be of no further use. Those who do go further will appreciate all the more the power of sophistication.

The presentation assumes a first acquaintance with general relativity, although we give a brief recapitulation of (some of) the main points in chapter 1. The treatment is however very incomplete: we do not consider the Einstein field equations because we do not demonstrate here that the black hole geometries are solutions of the equations. In fact, very little prior knowledge of relativity is required to study the properties of given black hole spacetimes. Chapter 2 is devoted to classical spherically symmetric, or non-rotating (Schwarzschild) black holes in the vacuum and chapter 3 to axially symmetric, or rotating (Kerr) ones. It is unfortunate that even simple calculations for Kerr black holes rapidly become algebraically complex. We have tried not to let this obscure the intriguing physics. We ask the reader to stick with it. After all, there may be a lot of it, but it is only elementary algebra. We give only a brief overview of charged black holes. These have played an important role as algebraically simpler models for many of the properties of rotating holes, and they are important as such in higher dimensions, but it is intrinsically difficult to maintain an interest in the physics of objects that probably do not exist, especially since we are going to treat the rotating holes in detail anyway.

In chapter 4 we attempt to explain the quantum properties of black holes without recourse to quantum field theory proper. The calculations here are less rigorous than in the rest of the book (probably a gross understatement) but many variations on the standard theme are now available in textbooks, reviews and lecture notes on the internet and we see no merit in repeating these. We hope our approach is more useful than a crash course in quantum field theory. It does, of course, assume a more than superficial understanding of standard quantum theory. Chapter 6 closes the book with a brief review of black hole astrophysics in so far as it is relevant to the observation of black holes. For this second edition we have added chapter 5 on wormhole metrics and time travel and a set of solutions to the problems. The new edition has also given us the opportunity to revise and clarify some of the text and problems and to add some new problems.

We are aware that we have omitted many contemporary topics in black hole physics, not least the properties of general black holes, perturbation of black holes and the role of black holes in string theories. We regard these as beyond the scope of the book (and in the last case of the expertise of the authors). We hope (and believe) that working out long-hand the details of what we do include will provide a firm foundation for those students who will go on to study such advanced topics and a firm understanding and appreciation of the properties of black holes for those who do not.

Derek Raine
Ted Thomas

Contents

of astronomical object, the active galactic nuclei (or AGNs), in which the central objects have normal densities but masses of the order of $10^8\ M_\odot$. This combination of mass and density make AGNs candidates for black holes.

In a relativistic theory of gravity, by definition, the local speed of light is always c, the speed of light in a vacuum in the absence of gravity. The Newtonian picture of the emitted light being slowed down and turned back by gravity is therefore not appropriate. However, we can get a truer picture from an application of the equivalence principle. The presence of a gravitational field introduces a relative acceleration between freely-falling frames of reference. The equivalence principle then leads to an approximate relation between time interval $d\tau$ measured at radius R from a body of mass M and the corresponding time interval $d\tau'$ measured at infinity (Will, 1993)

$$d\tau \approx \left(1 - \frac{2GM}{Rc^2}\right)^{1/2} d\tau'. \tag{1.2}$$

(The exact relation depends on the full theory of gravity.) This says that a clock at radius r runs slow compared to a clock at infinity. There is a corresponding redshift z of light emitted at frequency ω and received at ω' given by $\omega/\omega' = 1 + z = d\tau'/d\tau$. This suggests that $z \to \infty$ as the Newtonian potential $2GM/R \to c^2$ and hence that the light from the surface of a body at this potential would be redshifted to invisibility. Thus the body would be a black hole. The relativistic condition $2GM/R = c^2$ is, of course, analytically the same as the Newtonian condition $v_{\rm esc} = c$. But the condition for the Newtonian approximation to be valid is that $GM/R \ll c^2$, so we require the full theory of gravity to investigate this behaviour consistently and to treat black holes correctly. Furthermore, the gravitational potential on the surface of a neutron star is about $0.1c^2$ so we need a general relativistic theory of stellar evolution to be confident of understanding evolution beyond this stage.

The replacement of Newton's theory of gravity by Einstein's general theory of relativity does not alter the relationship between mass and density in equation (1.1), except that now the density is to be interpreted as the average density within the boundary of the (non-rotating) black hole. But it does alter our picture of the spacetime of a black hole and how it gravitates. A relativistic black hole has no material surface; all of its matter has collapsed into a singularity that is surrounded by a spherical boundary called its event horizon. The event horizon is a one-way surface: particles and light rays can enter the black hole from outside but nothing can escape from within the horizon of the hole into the external universe. An outgoing photon that originates outside the event horizon can propagate to infinity but in so doing it suffers a gravitational redshift: in Newtonian language it looses energy in doing work against the gravitational potential. This redshift is larger the closer the point of emission is to the horizon. On the other hand a photon or particle emitted inside the horizon in any direction must inevitably encounter the singularity and be annihilated. (This is strictly true only in the simplest type of black hole: in more general black holes destruction is not inevitable and the fate of a particle or photon inside the hole is more complicated.) A photon emitted at the horizon towards a distant observer stays there indefinitely. For this reason one can think of the horizon as made up of outwardly directed photons.

1.2 Why study black holes?

The importance of black holes for gravitational physics is clear: their existence is a test of our understanding of strong gravitational fields, beyond the point of small corrections to Newtonian physics, and a test of our understanding of astrophysics, particularly of stellar evolution. Current theories show that black holes are an almost inevitable consequence of the way that massive stars evolve: we therefore expect to find black holes amongst the stars in the Galaxy, and it appears that we do.

There is also a surprising and quite unexpected reason why black holes turn out to be important: this is for the potential insight they offer into the connection between quantum physics and gravity. We shall see that black holes appear formally to satisfy the laws of thermodynamics, with Mc^2 in the role of internal energy, the acceleration due to gravity in the guise of temperature and the black hole area as entropy. But this turns out to be more than a formal analogy. When we include the effects of quantum physics we find that black holes behave as real objects with a non-zero temperature and entropy: in particular they radiate like black bodies. The analogy with thermodynamics is therefore not just a formal one, but black holes really do obey the laws of thermodynamics.

We can now turn this argument around. Since black hole radiation involves a mixture of gravity and quantum physics this connection necessarily leads us into the territory of quantum gravity, and, since quantum gravity is the missing link in a complete picture of the fundamental forces, to 'theories-of-everything'. Any theory-of-everything has to be consistent with thermodynamics, and hence with black hole thermodynamics. Therefore any theory-of-everything should be able to predict the thermodynamic properties of black holes from *ab initio* statistical calculations. It is therefore interesting that theories that treat 'strings' as fundamental entities have been partially successful in this regard. It appears that black holes will play a central role in our understanding of fundamental physics.

1.3 Elements of general relativity

It is assumed that the reader has had a first acquaintance with a course on general relativity, for example from one of the many excellent introductory textbooks (for example, Kenyon, 1990, Hartle, 2003). In this section we shall present some of the main ideas of the theory, but only in the form of a brief review.

1.3.1 The principle of equivalence

The principle of equivalence tells us that local experiments (those carried out in the immediate vicinity of an event) cannot distinguish between an accelerated frame of reference and the presence of a gravitational field. In both cases we observe that bodies subject to no non-gravitational forces fall with equal acceleration. (We use the double negative in 'no non-gravitational forces' to emphasise that gravity may or may not be present, but no other forces are acting.) This creates difficulties for the Newtonian approach to dynamics because that requires us to choose a non-accelerated (or 'inertial') frame of reference. In Newtonian physics we get round this problem by designating the distant stars as a non-accelerated reference frame. This is a non-local, non-causal solution and therefore unsatisfactory. It is obviously non-local, and it is non-causal because there is no mechanism by which this reference frame is singled out, except by the fact that it gives the right answers (for example, for the motion of the planets in the Solar System).

Einstein was struck by the observation that all bodies fall with the same acceleration in a gravitational field. Newtonian gravity offers no explanation for this *universality of free-fall.* So Einstein used the universality of free fall to enunciate the principle of equivalence and made this the basis of his general relativistic theory of gravity. The principle of equivalence implies that in free fall we cannot detect the presence of a gravitational field by local experiments. (In free fall all bodies move inertially whether or not gravity is present.) Therefore in a local freely falling frame of reference we already know the laws of physics in the presence of gravity: they are the same as if gravity were absent!

1.3.2 The Newtonian affine connection

From a practical point of view it is not very easy to use the principle of equivalence directly for calculations. This is because in the presence of gravity the local freely falling frame of reference is changing from event to event and we are not experienced at doing calculations in ever changing frames of reference. Rather we need to translate this point of view into a fixed, but arbitrary, reference frame. Although reference frames and coordinate systems are not the same thing, (because the axes of a reference frame are not required to be tangents to coordinate lines), for the present purposes we shall ignore the distinction between them.

Imagine therefore that the system of coordinates $(\xi^0, \xi^1, \xi^2, \xi^3) = (\xi^\mu)$, $(\mu = 0, 1, 2, 3)$ corresponds momentarily to the natural choice of the freely-falling observer, with ξ^0 the Newtonian time (up to a factor of c). Suppose further that another set of coordinates (x^μ) are defined in a global patch (although not necessarily the whole) of spacetime. Each set of coordinates is given in terms of the other by $x^\mu = x^\mu(\xi^\nu)$ and $\xi^\mu = \xi^\mu(x^\nu)$, $(\mu, \nu = 0, 1, 2, 3)$. Note that on the left of these equations the variable stands for an independent coordinate and on the right for a function. The eliding of these distinct meanings by use of the same symbol is common practice and useful for keeping track of dependencies provided that care is taken.

According to Newtonian physics a test body subject to no non-gravitational forces will move along a spacetime trajectory defined by

$$\frac{d^2\xi^\mu}{d\tau^2} = 0. \tag{1.3}$$

Transforming to our (x^μ) coordinates this becomes, after some calculation of partial derivatives,

$$\frac{d^2x^\mu}{d\tau^2} + \frac{\partial x^\mu}{\partial \xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\nu \partial x^\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0, \tag{1.4}$$

where we are employing the usual summation convention implying a sum over repeated indices. The Greek indices are assumed to range over the values 0,1,2,3 throughout. To derive (1.4) from Eq. (1.3) we have used

$$\frac{\partial x^\alpha}{\partial \xi^\mu}\frac{\partial \xi^\mu}{\partial x^\beta} = \delta^\alpha_\beta, \tag{1.5}$$

which follows from differentiation of $x^\alpha(\xi^\mu(x^\beta)) = x^\alpha$. Eq. (1.4) is of the form

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0. \tag{1.6}$$

This equation for the motion of a test body holds whether or not gravity is present. If gravity is present the only place it can appear in this equation is through the quantity $\Gamma^\mu_{\nu\rho}$, called the affine connection. Looked at in this way, gravity therefore does not enter through an additional force term on the right hand side of (1.3). The only difference between the presence or absence of a gravitational field is that in the latter case it will be possible to recover the Eq. (1.3) everywhere in a single inertial coordinate system, not just locally in a freely-falling frame (because in the absence of gravity a local freely-falling frame is automatically a global inertial frame).

1.3.3 Newtonian gravity

To make the connection with the usual form of the equation of motion of a test body in Newtonian gravity we must be able to choose coodinates in which the affine connection takes an appropriate form. In fact, we must have, in some suitably chosen frame of reference (x^μ),

$$\Gamma^0_{ij} = 0; \quad \Gamma^i_{0j} = 0; \quad \Gamma^i_{00} = \frac{\partial \phi}{\partial x^i},$$

where $i, j = 1, 2, 3$, and ϕ is the Newtonian gravitational potential, since with these values for $\Gamma^\mu_{\nu\rho}$ we recover from (1.5) the equations of motion in a Newtonian gravitational field:

$$x^0 = ct = c\tau;$$

$$\frac{d^2x^i}{d\tau^2} = -\frac{\partial \phi}{\partial x^i}.$$

This also tells us that the affine connection is related to the distribution of matter through the extension to a general coordinate system of Poisson's equation $\nabla^2\phi = 4\pi G\rho$. Since this involves second derivatives of ϕ the relation between the affine connection and matter must involve the derivatives of the affine connection. The appropriate combinations of derivatives can be shown to be related to the curvature of (Newtonian) spacetime.

Newtonian gravitation is therefore a theory of the structure of spacetime, the relevant structures being the affine connection, the privileged time coordinate t and the Euclidean spatial metric. In Newtonian physics the (affine) geometry of spacetime is measured by the paths of particles and is unrelated to the geometry of time and space as measured by clocks and rods. Relativity is a lot simpler: there is only one geometry. The geometry of time and space, as measured by clocks and rods, itself governs the motion of particles, as we shall explain below.

1.3.4 Metrics in relativity

In a freely-falling frame special relativity is valid *locally* (whether or not gravity is present), and the spacetime interval (the proper distance or proper time) between neighbouring events, ds, is given by the familiar line element (or metric)

$$ds^2 = c^2dt^2 - (dx^2 + dy^2 + dz^2), \tag{1.7}$$

where t, x, y, z are the time and rectangular spatial coordinates of the freely falling (inertial) observer. In tensor notation this line element can be written

$$ds^2 = \eta_{\alpha\beta}d\xi^\alpha d\xi^\beta, \tag{1.8}$$

where $\eta_{\alpha\beta}$ is the metric tensor and we are using $(\xi^\alpha) = (ct,\ x,\ y,\ z)$ for this special coordinate system at a point. With these coordinates the metric has components $\eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1)$. In this convention ds^2 is positive for a timelike interval, zero for a lightlike interval and negative for a spacelike interval. This is convenient for dealing with the motion of particles. For positive ds^2 then $ds/c = d\tau$, where $d\tau$ is the proper time between the events. For negative ds^2 then $(-ds^2)^{1/2} = dl$ is the proper distance between the two events. An alternative convention often used for the metric is $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$. Care is needed in copying formulae from references to adjust these, if necessary, to the convention being employed.

In a general coordinate system, which we shall call (x^μ), we have

$$ds^2 = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}dx^\mu dx^\nu = g_{\mu\nu}dx^\mu dx^\nu \tag{1.9}$$

(from the chain rule for partial derivatives, $d\xi^\alpha = (\partial\xi^\alpha/\partial x^\mu)dx^\mu$). The quantities $g_{\mu\nu}$ are called the metric coefficients. They are symmetric, that is to say $g_{\mu\nu} = g_{\nu\mu}$, and are, in general, functions of the spacetime coordinates.

In the absence of gravity we can find a *global* coordinate system (ξ^α) in which the metric takes the form (1.8) everywhere. In the presence of gravity we can find such coordinates only in an infinitesimal neighbourhood of each spacetime point.

Problem 1 *Show that neither the Minkowski metric Eq. (1.7) nor the sum of squares is an invariant under the Galilean transformation and hence that there is no interval in Newtonian spacetime (i.e. Newtonian spacetime does not admit a metric).*

The motion of particles is again governed by Eq. (1.6) in relativity, because the arguments leading to it are still valid, except that now both $t \neq \tau$ and $x^0 \neq c\tau$ (since these would not be compatible with the metric Eq. (1.8) or (1.9)).

Now we see that in local freely falling coordinates the metric is obtained from first derivatives of the ξ^μ whereas the affine connection depends on second derivatives. We therefore expect that there will exist a relation between the components of the affine connection and derivatives of components of the metric. This is indeed the case:

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} \left(\partial_\nu g_{\rho\alpha} + \partial_\rho g_{\nu\alpha} - \partial_\alpha g_{\nu\rho} \right), \tag{1.10}$$

where $\partial_\alpha f = \partial f / \partial x^\alpha$ and $(g^{\mu\nu})$ is the inverse matrix to $(g_{\mu\nu})$, so

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu.$$

Thus we find the values of $\Gamma^\mu_{\nu\rho}$ from the values of the metric coefficients appropriate to a particular gravitational field.

So in relativity there is just one spacetime geometry defined by the metric and measured by both clocks and rods *and* by particle paths. The motion of particles, is governed by Eq. (1.5), called the geodesic equation (because it is of the same form as the equation for the shortest paths on a curved surface, which are called geodesics). Note that the physical interpretation of the coordinates (x^μ) (what they measure physically) depends on the metric coefficients, so cannot be determined unless the metric is known.

1.3.5 The velocity and momentum 4-vector

In this section we define two important 4-vectors relating to the motion of a particle, namely its velocity and momentum; acceleration will be dealt with later. The 4-velocity vector of a particle with position $(x^\alpha(\tau))$ is given by

$$(u^\mu) = \left(\frac{dx^\mu}{d\tau} \right) = \left(\frac{dx^0}{d\tau}, \frac{dx^1}{d\tau}, \frac{dx^2}{d\tau}, \frac{dx^3}{d\tau} \right),$$

formally as in special relativity. The 4-momentum of a particle of mass m_0 is $p^\mu = m_0 u^\mu$.