CALCULUS
OF SEVERAL VARIABLES
SECOND EDITION

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PREFACE

Like the first edition that preceded it, this second edition of *Calculus of Several Variables* is intended for students in science and engineering who have already completed a study of the techniques and applications of differential and integral calculus of real-valued functions of a single real variable. Thus it is designed for the third and fourth semesters of a two-year calculus program. Typically, the third semester deals with partial differentiation and multiple integration, and the fourth with vector-valued functions.

Although this book was written as a sequel to the author's *Single-Variable Calculus, 2nd Edition* (Addison-Wesley, 1990), it does not require previous use of that book, nor makes specific reference to it. Most of this book corresponds to the last seven chapters of the author's *Calculus: A Complete Course* (Addison-Wesley, 1990). Chapters 1 to 7 of this book are essentially identical to Chapters 12 to 18 of the *Complete Course*; all sections and exercises are the same except that this book has one extra section in Chapter 7 devoted to orthogonal curvilinear coordinates. Also, the four appendices in this book present, respectively, material on two-dimensional vectors, conic sections, first-order differential equations, and second-order differential equations, all of which have been drawn from various chapters and appendices of *Calculus: A Complete Course*.

Although this book treats the same topics as the first edition, and largely in the same order, many local revisions have been made to clarify explanations, to improve exercise sets (by additions, deletions, and reorderings), and to provide some extra material. Some sections, such as those dealing with parametric surfaces and surface integrals, have been completely rewritten. A new section (7.6) develops formulas for the vector operators **grad**, **div**, and **curl** in terms of general orthogonal curvilinear coordinates. This was added at the request of some students who needed that material in electrical engineering courses.

Some Features of the Text

- There is an emphasis on geometry and the use of geometric reasoning in solving problems.
- Wherever appropriate the relationship between several-variable calculus and linear algebra is noted. For example, the Chain Rule is interpreted in terms of matrix multiplication. A brief, optional section on matrix algebra is included in the first chapter for this purpose.

- Partial differential equations, in particular Laplace's equation, the heat (diffusion) equation, and the wave equation, are used to illustrate elementary calculations involving partial derivatives.
- Numerous applications of multivariable calculus are included in the book. Some, such as the method of least squares, perturbation methods, envelopes of families of curves and surfaces, and Newton's method for systems of equations, are not always to be found in calculus books.
- Classical mechanics is emphasized, especially in the applications of multiple integrals, line and surface integrals. Such applications are facilitated by constantly regarding integrals as "sums" of elements.
- Vector methods can greatly simplify the solution of problems in classical mechanics. As an example, Kepler's laws of planetary motion are derived via elementary vector calculations rather than by the usual method of making auspicious (but obscure) changes of variables in the appropriate differential equation.
- Chapter 5 deals with the differential geometry of general curves in 3-space, developing the Frenet-Serret formulas and showing that the shape of a curve is determined by its curvature and torsion functions. Rotating frames of reference (the Coriolis effect) is also discussed.
- The fundamental theorems of calculus in 3-space (Stokes's Theorem and Gauss's Divergence Theorem) are applied to problems in fluid mechanics, electrostatics and magnetostatics.
- The properties of conic sections are developed in Appendix 2, partly by analytic methods and partly by elementary geometric arguments. Polar equations of conics are treated in Section 5.4 on Kepler's Laws.

Core and Optional Material

Any division of material into "core" and "optional" is necessarily somewhat arbitrary. Most instructors would agree that the material of Sections 1.1–1.4, 2.1–2.6, 4.1–4.5, 5.1–5.2, 6.1 — 6.6, and 7.1–7.4 constitute a core course in multivariable and vector calculus. However, most of us would also be loath to teach a course devoid of applications so we would certainly want to include part of Chapter 3, say Sections 3.1–3.3, and at least parts of Sections 4.6 and 7.5. Some of the most interesting applications are in Sections 5.3 and 5.4, and they should be included in any course where time permits or need for the topics requires. Each of Sections 3.4–3.6 is self-contained and optional. Section 3.6 contains the only strictly numerical topic, Newton's method. Even for functions of only two variables, multivariable numerical methods can be sufficiently complicated that the student should have a sophisticated programmable calculator or, preferably, a computer to use them.

Acknowledgments

The first edition of *Calculus of Several Variables* has been used, since its publication, for classes of engineering and science students at the University of British Columbia. I am grateful to colleagues and students at UBC, and at other institutions where the book has been used, for their encouragement and useful comments and criticisms.

Especially helpful comments were made by Jim Carrell and John Walsh at UBC. Many of the changes in this edition are a result of that feedback. I am also very grateful to Ken MacKenzie (McGill University) who undertook a thorough checking of the Answers section.

I typeset this volume using TEX and PostScript on an AT microcomputer. I also generated most of the figures in PostScript using the *MG System*, a PC-based mathematical graphics software package developed by my colleague Professor Robert Israel and myself. Some of the three dimensional air-brush art was prepared by Iris Ward.

I wish to thank several people at Addison-Wesley for their assistance and encouragement. These include Sponsoring Editor Jim Grant, who guided the three-volume project of which this is the final installment, Vice-President Andy Yull, with whom I enjoyed stimulating discussions on matters of design and on problems involving the TEX–PostScript interface, and especially Executive Editor Ron Doleman who has supervised the publication of several of my books, and who first introduced me to TEX and PostScript.

While I have tried, with much excellent help, to make this second edition as free from errors and obscurities as possible, I am not so naïve as to believe that there are none left. Any comments, corrections, and suggestions for future revisions from readers will be much appreciated.

R.A.A.
Vancouver, Canada
January, 1991

CONTENTS

CHAPTER 1

Coordinate Geometry and Vectors in 3-Space

A complete real-variable calculus program involves the study of

i) real-valued functions of a single real variable,

ii) real-valued functions of a real vector variable,

iii) vector-valued functions of a single real variable,

iv) vector-valued functions of a real vector variable.

Item (i) is the subject of a single-variable calculus course; we assume that the reader is already familiar with that subject. This book is about items (ii), (iii), and (iv). Specifically, Chapters 2–4 are concerned with the differentiation and integration of real-valued functions of several real variables, that is, of a real vector variable. Chapter 5 and part of Chapter 6 deal with vector-valued functions of a single real variable. Most of Chapters 6 and 7 present aspects of the calculus of functions whose domains and ranges both have dimension greater than one, that is, vector-valued functions of a vector variable. Mostly we will limit our attention to vector functions in two- and three-dimensional space.

In this chapter we will lay the foundation for multi-variable calculus by discussing analytic geometry and vectors in three and more dimensions. We assume that the student, having already undertaken a study of single-variable calculus, is familiar with the coordinate geometry of the Cartesian plane. We also introduce matrices in spaces of any dimension, as these are useful (but not essential) for formulating some of the concepts of calculus. This chapter is not intended to be a course in linear algebra. We develop only those aspects which we will use in later chapters, and omit most proofs.

1.1 ANALYTIC GEOMETRY IN THREE AND MORE DIMENSIONS

We say that the physical world in which we live is three dimensional because through any point there can pass three, and no more, straight lines which are **mutually perpendicular**, that is to say, each of them is perpendicular to the other two. This is equivalent to the fact that we require three numbers to locate a point in space with respect to some reference point (the *origin*). One way to use three numbers to locate a point is by having them represent (signed) distances from the origin, measured in the directions of three mutually perpendicular lines passing through the origin. We call such a set of lines a Cartesian coordinate system, and each of the lines is called a coordinate axis. We shall usually call these axes the x-axis, the y-axis and the z-axis, regarding the x- and y-axes as lying in a horizontal plane and the z-axis as vertical. Moreover, the coordinate system should have a **right-handed orientation**. This means that the thumb, forefinger and middle finger of the right hand can be extended so as to point respectively in the directions of the positive x-axis, the positive y-axis and the positive z-axis. For the more mechanically minded, a right-handed screw will advance in the positive z direction if twisted in the direction of rotation from the positive x-axis towards the positive y-axis. (See Fig. 1.1.1.)

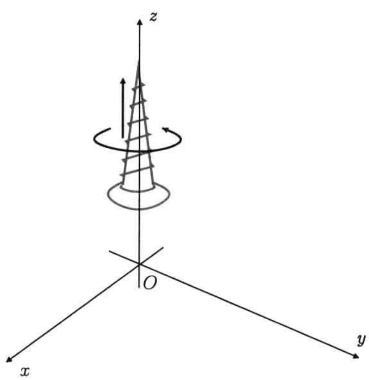

FIGURE 1.1.1

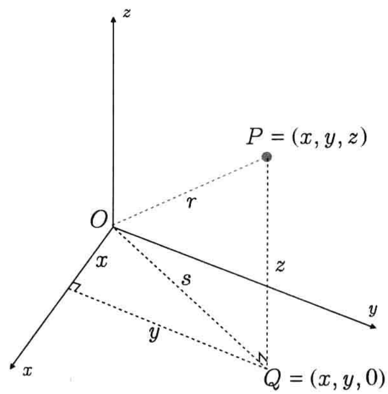

FIGURE 1.1.2

With respect to such a Cartesian coordinate system, the *coordinates* of a point P in 3-space constitute an ordered triple of real numbers, (x, y, z). The numbers x, y and z are, respectively, the signed distances of P from the origin, measured in the directions of the x-axis, the y-axis and the z-axis. (See Fig. 1.1.2.)

Let Q be the point with coordinates $(x, y, 0)$. Then Q lies in the xy-plane directly under (or over) P. (Q is the vertical projection of P onto the xy-plane.) If r is the distance from the origin O to P and s is the distance from O to Q, then, using two right-angled triangles, we have

$$s^2 = x^2 + y^2 \qquad \text{and} \qquad r^2 = s^2 + z^2 = x^2 + y^2 + z^2.$$

Thus the distance from P to the origin is given by

$$r = \sqrt{x^2 + y^2 + z^2}.$$

Similarly, the distance between points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Just as the x- and y-axes divide the xy-plane into four quadrants so also the three **coordinate planes** in 3-space (the xy-plane, the xz-plane and the yz-plane) divide 3-space into eight **octants**. We call the octant in which $x \geq 0$, $y \geq 0$ and $z \geq 0$ the **first octant**. When we draw graphs in 3-space it is sometimes easier to draw only the part lying in the first octant.

Equations and inequalities involving the three variables x, y and z generally define subsets of points in 3-space. Usually a single equation represents a surface (a two-dimensional object), and a single inequality represents a three-dimensional region (having volume). Two equations represent the intersection of the two surfaces represented by each of them, and so usually represent a curve or line (a one-dimensional object).

EXAMPLE 1.1.1

i) The equation $y = 0$ is satisfied by those points, and only those points, which lie in the xz-plane, that is, the vertical plane containing the x- and z-axes. Thus $y = 0$ is the equation of that plane.

ii) The equation $z = 1$ represents the horizontal plane consisting of all points lying at distance one unit above the xy-plane.

iii) The inequality $z \geq 1$ represents the *half-space* consisting of all points lying on or above the plane in (ii).

iv) The equation $y = x$ represents the vertical plane (the plane parallel to the z-axis) passing through the line with equation $y = x$ in the xy-plane. (See Fig. 1.1.3.)

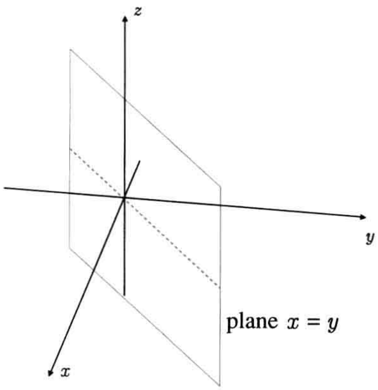

FIGURE 1.1.3

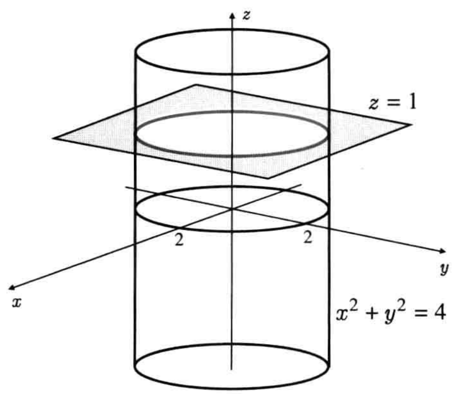

FIGURE 1.1.4

v) The equation $x^2 + y^2 + z^2 = 4$ represents the *sphere* consisting of all points at distance 2 from the origin.

vi) The inequality $x^2 + y^2 + z^2 \leq 4$ represents the *ball* consisting of all points inside or on the sphere in (v).

vii) In the xy-plane the equation $x^2 + y^2 = 4$ represents a circle of radius 2 centred at the origin. In 3-space it represents the circular cylinder with axis along the z-axis which intersects the xy-plane in that circle. (See Fig. 1.1.4.) Since the equation does not depend on z, all points on the vertical line through any point on the surface will also lie on the surface.

viii) The pair of equations $x^2 + y^2 = 4$, $z = 1$ represents the circle in which the horizontal plane $z = 1$ intersects the vertical cylinder $x^2 + y^2 = 4$. The circle has radius 2 and centre at the point $(0, 0, 1)$. (See Fig. 1.1.4.)

In Section 1.4 we will see many more examples of geometric objects in 3-space represented by simple equations.

Euclidean n-Space

Mathematicians and users of mathematics frequently need to consider **n-dimensional space** where n is greater than three, and may even be infinite. Students sometimes have difficulty in visualizing a space of dimension four or higher. The secret to dealing with these spaces is to regard the points in n-space as *being* ordered n-tuples of real numbers; that is, $(x_1, x_2, \dots, x_n)$ is a point in n-space instead of just being the coordinates of such a point. We stop thinking of points as existing in physical space and start thinking of them as algebraic objects. We usually denote n-space by the symbol $\mathbb{R}^n$ to show that its points are n-tuples of *real* numbers. Thus $\mathbb{R}^2$ and $\mathbb{R}^3$ denote the plane and 3-space respectively. Note that in passing from $\mathbb{R}^3$ to $\mathbb{R}^n$ we have altered the notation a bit — in $\mathbb{R}^3$ we called the coordinates x, y and z while in $\mathbb{R}^n$ we called them x_1, x_2, ... and x_n so as not to run out of letters. We could, of course, talk about coordinates (x_1, x_2, x_3) in $\mathbb{R}^3$, and (x_1, x_2) in the plane $\mathbb{R}^2$, but (x, y, z) and (x, y) are traditionally used there.

While we think of points in $\mathbb{R}^n$ as n-tuples rather than geometric objects, we do not want to lose all sight of the underlying geometry. By analogy with the two- and three-dimensional cases, we still consider the quantity

$$\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}$$

as representing the *distance* between the points with coordinates $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$. Also, we call the $(n-1)$-dimensional set of points in $\mathbb{R}^n$ which satisfy the equation $x_n = 0$ a "hyperplane" by analogy with the plane $z = 0$ in $\mathbb{R}^3$.

EXERCISES

Find the distance between the pairs of points in Exercises 1–4.

1. $(0, 0, 0)$ and $(2, -1, -2)$ **2.** $(-1, -1, -1)$ and $(1, 1, 1)$

3. $(1, 1, 0)$ and $(0, 2, -2)$ **4.** $(3, 8, -1)$ and $(-2, 3, -6)$

5. What is the shortest distance from the point (x, y, z) to a) the xy-plane? b) the x-axis?

6. Show that the triangle with vertices $(1, 2, 3)$, $(4, 0, 5)$ and $(3, 6, 4)$ has a right angle.

7. Find the area of the triangle with vertices $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$.

8. What is the distance from the origin to the point $(1, 1, \dots, 1)$ in $\mathbb{R}^n$?

9. What is the distance from the point $(1, 1, \dots, 1)$ in n-space to the closest point on the x_1-axis?

In Exercises 10–21 describe (and sketch if possible) the set of points in $\mathbb{R}^3$ which satisfy the given equation or inequality.

10. $z = 2$ **11.** $y \geq -1$

12. $z = x$ **13.** $x + y = 1$

14. $x^2 + y^2 + z^2 = 4$

15. $(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = 4$

16. $x^2 + y^2 + z^2 = 2z$ **17.** $x^2 + y^2 \leq 4$

18. $x^2 + z^2 = 4$ **19.** $z = y^2$

20. $z \geq \sqrt{x^2 + y^2}$ **21.** $x + y + z = 1$

In Exercises 22–31 describe (and sketch if possible) the set of points in $\mathbb{R}^3$ which satisfy the given pair of equations or inequalities.

22. $\begin{cases} x = 1 \\ y = 2 \end{cases}$ **23.** $\begin{cases} x = 1 \\ y = z \end{cases}$

24. $\begin{cases} x^2+y^2+z^2=4 \\ z=1 \end{cases}$

25. $\begin{cases} x^2+y^2+z^2=4 \\ x^2+y^2+z^2=4x \end{cases}$

26. $\begin{cases} x^2+y^2+z^2=4 \\ x^2+z^2=1 \end{cases}$

27. $\begin{cases} x^2+y^2=1 \\ z=x \end{cases}$

28. $\begin{cases} y \ge x \\ z \le y \end{cases}$

29. $\begin{cases} x^2+y^2 \le 1 \\ z \ge y \end{cases}$

30. $\begin{cases} x^2+y^2+z^2 \le 1 \\ \sqrt{x^2+y^2} \le z \end{cases}$

31. $\begin{cases} z \ge x^2 \\ x^2+y^2+z^2 \le 1 \end{cases}$

1.2 VECTORS IN 3-SPACE

A vector is a quantity possessing both magnitude and direction. In this section we are assuming that you have encountered vectors in your previous mathematical studies. A brief introduction to two-dimensional vectors is given in Appendix 1, where they are used to describe the motion of an object in the plane $\mathbb{R}^2$. It is a good idea to read the first four pages of that appendix before proceeding further with this section. The algebra and geometry of plane vectors described there extends to spaces of any number of dimensions; we can still think of vectors as represented by arrows, and sums and scalar multiples are formed just as for plane vectors.

In this section we state properties of vectors, as developed in Appendix 1, for vectors in 3-space. It will be evident how extensions can be made to higher dimensions.

Given a Cartesian coordinate system in 3-space, we define three **standard basis vectors**, **i**, **j**, and **k**, represented by arrows from the origin to the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ respectively. See Fig. 1.2.1. Any vector in 3-space can be written as a *linear combination* of these basis vectors; for instance, the vector **r** from the origin to the point (x,y,z) is given by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

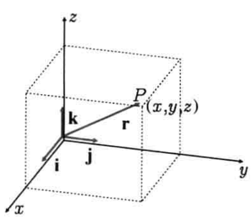

FIGURE 1.2.1

We say that **r** has **components** x, y, and z. The length of **r** is

$$|\mathbf{r}| = \sqrt{x^2+y^2+z^2}.$$

Such a vector from the origin to a point is called the **position vector** of that point. Generally, however, vectors do not have any specific location; two vectors are regarded as being equal if they have the same length and the same direction, that is, if they have the same components.

If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are two points in 3-space, then the vector $\mathbf{v} = \overrightarrow{P_1P_2}$ from P_1 to P_2 has components $x_2 - x_1$, $y_2 - y_1$, and $z_2 - z_1$, and is therefore represented in terms of the standard basis vectors by

$$\mathbf{v} = \overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$

Sums and scalar multiples of vectors are easily expressed in terms of components. If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and if t is a scalar (i.e. a real number) then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1)\mathbf{i} + (u_2 + v_2)\mathbf{j} + (u_3 + v_3)\mathbf{k},$$
$$t\mathbf{u} = (tu_1)\mathbf{i} + (tu_2)\mathbf{j} + (tu_3)\mathbf{k}.$$