Graduate Texts in Mathematics

132

Iteration of Rational Functions Complex Analytic Dynamical Systems

Springer-Verlag

Alan F. Beardon

Iteration of Rational Functions

Complex Analytic Dynamical Systems

With 35 Illustrations

Springer-Verlag
New York Berlin Heidelberg London Paris
Tokyo Hong Kong Barcelona Budapest

Alan F. Beardon
Department of Pure Mathematics
and Mathematical Statistics
University of Cambridge
Cambridge CB2 1SB
UK

Library of Congress Cataloging-in-Publication Data
Beardon, Alan F.
Iteration of rational functions/Alan Beardon.
p. cm. — (Graduate texts in mathematics; 132)
Includes bibliographical references and indexes.
ISBN 0-387-97589-6
1. Iterative methods (Mathematics) 2. Mappings (Mathematics)
I. Title. II. Series.
QA297.8.B43 1991
511'.4—dc20 91-24082

Printed on acid-free paper.

Typset by Asco Trade Typesetting Ltd., Hong Kong.
Printed and bound by R.R. Donnelley & Sons, Harrisonburg, VA.
Printed in the United States of America.

9 8 7 6 5 4 3 2 1

ISBN 0-387-97589-6 Springer-Verlag New York Berlin Heidelberg
ISBN 3-540-97589-6 Springer-Verlag Berlin Heidelberg New York

Alan F. Beardon

Iteration of Rational Functions

Springer-Verlag

Graduate Texts in Mathematics 132

Graduate Texts in Mathematics

1 TAKEUTI/ZARING. Introduction to Axiomatic Set Theory. 2nd ed.
2 OXTOBY. Measure and Category. 2nd ed.
3 SCHAEFFER. Topological Vector Spaces.
4 HILTON/STAMMBACH. A Course in Homological Algebra.
5 MAC LANE. Categories for the Working Mathematician.
6 HUGHES/PIPER. Projective Planes.
7 SERRE. A Course in Arithmetic.
8 TAKEUTI/ZARING. Axiometic Set Theory.
9 HUMPHREYS. Introduction to Lie Algebras and Representation Theory.
10 COHEN. A Course in Simple Homotopy Theory.
11 CONWAY. Functions of One Complex Variable. 2nd ed.
12 BEALS. Advanced Mathematical Analysis.
13 ANDERSON/FULLER. Rings and Categories of Modules.
14 GOLUBITSKY GUILEMIN. Stable Mappings and Their Singularities.
15 BERBERIAN. Lectures in Functional Analysis and Operator Theory.
16 WINTER. The Structure of Fields.
17 ROSENBLATT. Random Processes. 2nd ed.
18 HALMOS. Measure Theory.
19 HALMOS. A Hilbert Space Problem Book. 2nd ed., revised.
20 HUSEMOLLER. Fibre Bundles. 2nd ed.
21 HUMPHREYS. Linear Algebraic Groups.
22 BARNES/MACK. An Algebraic Introduction to Mathematical Logic.
23 GREUB. Linear Algebra. 4th ed.
24 HOLMES. Geometric Functional Analysis and Its Applications.
25 HEWITT/STROMBERG. Real and Abstract Analysis.
26 MANES. Algebraic Theories.
27 KELLEY. General Topology.
28 ZARISKI/SAMUEL. Commutative Algebra. Vol. I.
29 ZARISKI/SAMUEL. Commutative Algebra. Vol. II.
30 JACOBSON. Lectures in Abstract Algebra I. Basic Concepts.
31 JACOBSON. Lectures in Abstract Algebra II. Linear Algebra.
32 JACOBSON. Lectures in Abstract Algebra III. Theory of Fields and Galois Theory.
33 HIRSCH. Differential Topology.
34 SPITZER. Principles of Random Walk. 2nd ed.
35 WERMER. Banach Algebras and Several Complex Variables. 2nd ed.
36 KELLEY/NAMIOKA et al. Linear Topological Spaces.
37 MONK. Mathematical Logic.
38 GRAUERT/FRITZSCHE. Several Complex Variables.
39 ARVESON. An Invitation to C^* -Algebras.
40 KEMENY/SNELL/KNAPP. Denumerable Markov Chains. 2nd ed.
41 APOSTOL. Modular Functions and Dirichlet Series in Number Theory. 2nd ed.
42 SERRE. Linear Representations of Finite Groups.
43 GILLMAN/JERISON. Rings of Continuous Functions.
44 KENDIG. Elementary Algebraic Geometry.
45 LOÈVE. Probability Theory I. 4th ed.
46 LOÈVE. Probability Theory II. 4th ed.
47 MOISE. Geometric Topology in Dimentions 2 and 3.

continued after index

To Francis, Jessica, Yvonne and Luke

Preface

This is not a book for the experts, nor is it written by one; it is a modest attempt to lay down the basic foundations of the theory of iteration of rational maps in a clear, precise, complete and rigorous way. The author hopes that those who wish to learn something about the subject will be able to do so from this book in a relatively painless way, and that it will serve as a starting point from which many recent, and much deeper, works can be tackled with some confidence.

The book begins, and ends, with a chapter consisting entirely of examples. In the first chapter, the examples are quite straightforward and are discussed from first principles without the advantage of any theoretical developments. Many readers will want to omit this chapter, but its purpose is two-fold. First, this subject is of interest to a large number of people not all of whom are mathematicians, and it is hoped that some of these readers will appreciate the more gentle start offered by this chapter; and second, in this chapter I illustrated most of the basic results of the theory in specific examples. The last chapter also consists entirely of examples but, by contrast, a claim about a particular example here demands as much formal verification as does the proof of a theorem. The primary purpose of these examples is, of course, to illustrate the theory developed earlier, but in addition to this, they have been chosen to show the variety of possibilities that can occur, and some at least go beyond those for which the computer-generated illustrations are now so familiar. For the convenience of readers, I have included an index of examples at the end of the text.

I have included a brief section at the beginning which describes some of the elementary topics that I shall assume the reader is familiar with. Other (more advanced) material is assumed at several other places in the text, but there, some explanation is merged with the general discussion. Each chapter starts with a summary outlining the main objectives in that chapter. There are, of course, occasions when I need to use more advanced results from other parts of mathematics, and where I have thought that a brief discussion of these would materially assist the reader I have included such a discussion in the text. Where I felt that it would not, I have relegated further discussion to an appendix to that chapter. Finally, and perhaps inevitably, I accept that some important items are omitted (most notably, the existence of Herman rings), but this is not in any sense meant to be a complete account of the subject.

It has been my objective to provide as much detail as seems appropriate for an average graduate student to understand the argument completely and without too much effort, and the criterion for the inclusion of detail has been whether or not I thought that it would assist the reader. In several places there is some minor repetition of material; this is simply an acknowledgement that most readers do not read (and authors do not write) books in the same order as their pages are numbered and so, on occasions, it is helpful to some readers to have this repetition. The greatest difficulty seemed to be in placing the material in a coherent order, and to avoid constantly changing from one topic to another as seems to happen so often in other accounts of the subject: I believe that I have been reasonably successful in this but, ultimately, it is for the reader to judge. I believe that important mathematical points should be stressed (even when they are mathematically trivial), and I have written this book in the belief that the onus lies with authors, not readers, to provide the details.

There are references given in the text, but I have not attempted to include references to all results, nor to trace the results back to the original source: indeed, given some of the informal, expository (and sometimes incomplete) accounts of the subject that exist, this would have sometimes been difficult, although, of course, almost all of the results originate with Fatou and Julia. There are no original illustrations in the text; the existing pictures are more than adequate for my purposes and I am grateful for those who have allowed me to use their illustrations.

In writing this text, I have had to learn the subject myself, and I have relied heavily on the help, encouragement and advice of many people. Noel Baker generously supplied me with notes for a course he gave, and as well as reading the manuscript, has responded willingly to a stream of questions (not all sensible) from me. Keith Carne has also read the manuscript, and has listened patiently and responded to the ideas and difficulties I have had, and his interest and support in this project has been most valuable. David Herron, Bruce Palka, Cliff Earle, Kari Hag, Pekka Koskela and Shanshuang Yang participated in a seminar which worked through a large portion of the manuscript and their comments and suggestions have led to a significant improvement in the text. Norbert Steinmetz provided one of the ideas in Chapter 7, and Fred

Gehring, as before, has been a great support. To all these people, and others who have helped in various ways, I offer my thanks. Of course, I take full responsibility for any errors that remain.

Cambridge, England
November 1990

Alan F. Beardon

Prerequisites

This section contains notation, terminology and some of the results that are taken for granted in the text. First, the notation. The real line, the complex plane and the extended complex plane are denoted by $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{C}_\infty$ respectively, and throughout the text, Δ denotes the unit disc in $\mathbb{C}$. For any set A, the closure, the boundary and the interior of A (all with respect to some underlying space X which will be clear from the context) are $\bar{A}$, ∂A and $\text{Int}(A)$, or A^0, respectively. For sets A and B, $A - B$ denotes the set difference (rather than $A \backslash B$ which I find visually unattractive); thus

$$A - B = \{x \in A: x \notin B\},$$

and the complement of A in X is $X - A$.

The symbol $\mapsto$ defines a function f (for example, $x \mapsto x^2$) as well as, of course, $f(x) = x^2$. Often, visual clarity is improved if brackets are omitted, so I use $f(x)$ and fx interchangeably. Likewise, if the composition $x \mapsto f(g(x))$ is defined, it is denoted by fg. These liberties allow one to inject a particular emphasis into a formula; for example, $f(gx)$ is to be thought of as the f-image of $g(x)$, while $fg(x)$ (the same point) is the fg-image of x. The composition of f with itself n times is the n-th iterate f^n of f, and $f^0 = I$, the identity map. As usual, both notations f', f'', and $f^{(n)}$ are used for the derivatives of f.

A small amount of complex analysis is taken for granted, roughly speaking that which would be covered in a first (and conventional) course in the subject. For example, we shall assume familiarity with the Maximum Modulus Theorem, Schwarz's Lemma and Rouché's Theorem. All of these results can be found in, for example, [3]. We say that f is a d-fold map of V onto W if, for every w in W, the equation $f(z) = w$ has exactly d solutions in V (counting multiple solutions by their multiplicity); for example, a polynomial of degree d is a d-fold map of $\mathbb{C}$ onto itself. If $d = 1$ we say the map is univalent, and at various points in the text we shall use Hurwitz's Theorem (that if a sequence univalent analytic maps f_n converge uniformly to f on a domain D, then f is either constant or univalent in D). This too can be found in [3].

Finally, we shall assume familiarity with the very basic ideas of metric spaces, namely those up to, say, uniform continuity, compactness and connectedness. We stress, however, that the material in Chapter 1 needs none of these ideas, and that some attempt has been made to match progression through the text with an assumption of increasing mathematical maturity.

Contents

CHAPTER 1

Examples

In this chapter we introduce some of the main ideas in iteration theory by discussing a variety of simple examples. The discussions involve only elementary mathematics, and our sole objective is to illustrate and stress those features that will be met in a general context later.

§1.1. Introduction

This book is about the repeated application, or *iteration*, of a rational function,

$$R(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m},$$

of a complex variable z. Specifically, we select a starting point z_0 in the complex plane $\mathbb{C}$ and then apply R repeatedly constructing, in turn, the points

$$z_0, z_1 = R(z_0), z_2 = R(z_1), \ldots.$$

In general, we denote the composition of two functions f and g by juxtaposition so fg is the function $z \mapsto f(g(z))$, and we allow ourselves to write either $fg(z)$ or $f(gz)$ depending on which of these we wish to emphasize. With this notation, $z_n = R^n(z_0)$, and by convention, $R^0 = I$, where I is the identity map.

Many questions now present themselves; for example, does the sequence z_n converge, or, better still, for which values of the initial point z_0 does the sequence z_n converge? If the sequence z_n does not converge, can we say anything else about its behaviour and, in any case, how robust are the answers to these questions under a small change in the initial point z_0? Instead of looking at the future progress of z_0, we can also look at its history as represented, say, by the sequence

$$\ldots, z_{-2}, z_{-1}, z_0,$$

where again $z_{n+1} = R(z_n)$. In general, for a given z_0 there will be several different possibilities for z_{-1}, even more for z_{-2}, and so on, so here there is a case for considering the totality of such sequences arising from a given point z_0.

We can gain a little insight immediately by making some elementary observations about fixed points. A point ζ is a *fixed point* of R if $R(\zeta) = \zeta$, and it is clear that such points must have a special role to play in the theory. Suppose now that for some choice of z_0, the sequence z_n converges to w. Then (because R is continuous at w)

$$w = \lim_{n\to\infty} z_{n+1} = \lim_{n\to\infty} R(z_n) = R\left(\lim_{n\to\infty} z_n\right) = R(w),$$

so w is a fixed point of R: thus *if* $z_n \to w$, *then* $R(w) = w$. For example, if

$$R(z) = z^2 - 4z + 6, \tag{1.1.1}$$

then, regardless of the choice of z_0, if the sequence z_n converges it can only converge to 2, 3 or ∞ (we will discuss ∞ later). As

$$R(z) - 2 = (z - 2)^2,$$

the reader can now find those z_0 for which $z_n \to 2$.

If the fixed point ζ of R lies in $\mathbb{C}$, then the derivative $R'(\zeta)$ is defined and we say that ζ is:

(1) an *attracting fixed point* if $|R'(\zeta)| < 1$;
(2) a *repelling fixed point* if $|R'(\zeta)| > 1$; and
(3) an *indifferent fixed point* if $|R'(\zeta)| = 1$.

This classification will be discussed again in much greater detail in Chapter 6, but it will be helpful to make some preliminary remarks now. If z is close to the fixed point ζ, then, *approximately*,

$$|R(z) - \zeta| = |R(z) - R(\zeta)| = |R'(\zeta)| . |z - \zeta|,$$

so points close to an attracting fixed point move even closer to it when we apply R, while points close to a repelling fixed point tend to move away from it. In particular, if z_0 lies sufficiently close to an attracting fixed point ζ, then $z_n \to \zeta$ as $n \to \infty$. On the other hand, if z is close to (but not equal to) a repelling fixed point ζ, initially it is repelled away from ζ, *but it may return to the vicinity of* ζ (or even to ζ itself) at a later stage. In fact, the only way that z_n can converge to a repelling fixed point ζ, is to have $z_n = \zeta$ for $n \geq n_0$, say. To see this, we suppose that $z_n \to \zeta$, where $z_n \neq \zeta$ for any n, and seek a contradiction. Certainly, the fact that the z_n converge to, but are distinct from, ζ implies that for infinitely many n,

$$|z_{n+1} - \zeta| < |z_n - \zeta|.$$