I. P. NATANSON

CONSTRUCTIVE FUNCTION THEORY

VOLUME I
UNIFORM APPROXIMATION

Translated by
ALEXIS N. OBOLENSKY

A volume, equally elegant in argumentation and in presentation, of one of the most interesting mathematical branches, dealing with approximate representation of arbitrary functions.

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INTRODUCTION

The constructive theory of functions is a branch of mathematical analysis dealing with questions that arise in the approximate representation of arbitrary functions by the simplest analytical expedients possible.

In this book we shall refrain from considering classes of functions of too extensive a character, and shall restrict ourselves to the investigation of the following two important classes:

I. Real functions that are defined and continuous over a specified segment $[a, b]$. We shall denote the set of all these functions by $C([a, b])$.

II. Real functions that are defined and continuous over the entire real axis $(-\infty, +\infty)$, and at the same time have the period 2π, so that the equation

$$f(x + 2\pi) = f(x)$$

is true for every value of x. We shall denote the set of all these functions by $C_{2\pi}$.

For the approximate representation of functions of both these classes we shall employ two particularly simple types of function: For class $C([a, b])$, the usual algebraic polynomials

$$P(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

with real coefficients; for class $C_{2\pi}$, however, the trigonometric polynomials, i.e. functions of the form

$$T(x) = A + (a_1 \cos x + b_1 \sin x) + \cdots + (a_n \cos nx + b_n \sin nx)$$

with real coefficients A, a_k, b_k.

We still have to explain what we mean by the approximate representation of a function $f(x)$ by a polynomial $P(x)$ or $T(x)$. This may be done in a number of ways.

We shall denote a polynomial $P(x)$ as an approximate function for a function $f(x) \in C([a, b])$, if *for all values of* $x \in [a, b]$ the inequality

$$|P(x) - f(x)| < \varepsilon$$

holds true, where the constant $\varepsilon > 0$ is characteristic of the degree of approximation attained.

Similarly in this connection we denominate a trigonometric polynomial $T(x)$ as an approximation for a function $f(x) \in C_{2\pi}$, if for *all real values of* x, the inequality

$$|T(x) - f(x)| < \varepsilon$$

holds true.

On account of the 2π periodicity of $T(x)$ it is, furthermore, sufficient if this last inequality be satisfied in a segment of length 2π, for instance, in the segment $[0, 2\pi]$, and even simply in the half-segment open to the right, since it is then actually satisfied over the whole of the axis.

If we make the principle of evaluation, thus defined, basic for the closeness of approximation in our theory, then we may denominate it as the *Theory of Uniform Approximation.*

Clearly from this standpoint, the value

$$\max_{a \leq x \leq b} |P(x) - f(x)|$$

may serve as the "measure" of the accuracy attained in the case $f(x) \in C[a, b]$, and the value

$$\max_{-\infty < x < +\infty} |T(x) - f(x)|$$

in the case $f(x) \in C_{2\pi}$. This is, so to speak, the "distance" between $f(x)$ and $P(x)$, or between $f(x)$ and $T(x)$, respectively.

[Parts II and III, described below, are to be published in English at a later date.]

Part II of the book is devoted to the *Theory of Mean Value Approximation.* We shall there deal with the approximate representation of functions $f(x)$ of an essentially more general type, for which we shall, however, again make use of the standard algebraic polynomials $P(x)$ and trigonometric polynomials $T(x)$ as approximate functions; we shall, nevertheless, change the criterion for the attained accuracy of the approximation.

We shall, in fact, use the integral

$$\int_a^b [P(x) - f(x)]^2 \, dx$$

as the "measure of distance" of the two functions $f(x)$ and $P(x)$, and similarly the integral

$$\int_{-\pi}^{\pi} [T(x) - f(x)]^2 \, dx$$

as the distance of a trigonometric polynomial $T(x)$ from a given function $f(x)$ in the segment $[-\pi, +\pi]$.

The approach thus modified will lead us to an essentially different theory with new formulations of problems and fresh results.

Finally in Part III, we shall investigate the problems arising out of *interpolation.* As a criterion of the approximation of a polynomial $P(x)$ to a function $f(x) \in C([a, b])$, neither the smallness of the value

$$\max_{a \leq x \leq b} |P(x) - f(x)|$$

nor of

$$\int_a^b [P(x) - f(x)]^2 \, dx$$

will be of service, but the actual coincidence of $P(x)$ with $f(x)$ at various previously selected points ("interpolation nodes")

$$x_1, x_2, \ldots, x_n$$

of the segment $[a, b]$. The same problem arises in the approximation of a function $f(x) \in C_{2\pi}$ by a trigonometric polynomial $T(x)$, in which the interpolation nodes must lie in one and the same segment of the length 2π.

As we shall see, all these points of view are most closely interconnected, so that the theories pertaining to them overlap to a very high degree. In fact, it is this interlocking of manifold ideas, methods, and facts that—quite apart from its highly practical significance—is chiefly responsible for the constructive theory of functions being one of the most beautiful branches of mathematics.

CONTENTS

UNIFORM APPROXIMATION

CHAPTER I

WEIERSTRASS' APPROXIMATION THEOREMS

§ 1. WEIERSTRASS' First Theorem

The following fundamental question is intrinsic in the theory of continuous approximation from the outset: "Can every arbitrary, continuous function in general be approximately represented by a polynomial with arbitrarily postulated accuracy?" WEIERSTRASS [1] [1] found it possible to give an affirmative answer to this in 1885. We formulate his result as:

WEIERSTRASS' First Approximation Theorem. *For an arbitrarily assumed $f(x) \in C([a, b])$, where $\varepsilon > 0$, a polynomial $P(x)$ exists of such type that for all values of $x \in [a, b]$, the inequality $P(x) - f(x) < \varepsilon$ is satisfied.*

From the large number of proofs now available for this theorem, I here present one that depends on another equally important theorem in analysis, i.e., one of S. N. BERNSTEIN's theorems [1].

Lemma 1. *The following identities hold good:*

$$\sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} = 1, \tag{1}$$

$$\sum_{k=0}^{n} (k - nx)^2 C_n^k x^k (1-x)^{n-k} = nx(1-x). \tag{2}$$

Proof. Identity (1) is trivial; it follows straight from the standard binomial formula, putting $a = x$ and $b = 1 - x$ in the expansion

$$(a+b)^n = \sum_{k=0}^{n} C_n^k a^k b^{n-k}. \tag{3}$$

The proof of the second identity is not quite so simple. Putting $a = z$, and $b = 1$, (3) above gives us the identity

$$\sum_{k=0}^{n} C_n^k z^k = (z+1)^n \tag{4}$$

Taking the first differential of (4) and multiplying it by z, we get

$$\sum_{k=0}^{n} k C_n^k z^k = nz(z+1)^{n-1}, \tag{5}$$

[1] Figures in square brackets refer to the bibliography at the end of this volume.

and differentiating (5) and again multiplying the result by z, we get

$$\sum_{k=0}^{n} k^2 C_n^k z^k = nz(nz+1)(z+1)^{n-2}. \tag{6}$$

Now put $z = \dfrac{x}{1-x}$ in the three identities (4), (5), and (6), and multiply each of them by $(1-x)^n$. This will give three new identities

$$\sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} = 1, \tag{7}$$

$$\sum_{k=0}^{n} k C_n^k x^k (1-x)^{n-k} = nx, \tag{8}$$

$$\sum_{k=0}^{n} k^2 C_n^k x^k (1-x)^{n-k} = nx(nx+1-x). \tag{9}$$

Multiplying (7), (8), and (9), respectively, by n^2x^2, $-2nx$, and 1, and adding the results, we obtain the required identity (2).

Corollary. *For all values of* x

$$\sum_{k=0}^{n} (k-nx)^2 C_n^k x^k (1-x)^{n-k} \leqq \frac{n}{4}. \tag{10}$$

For, since $$4x^2 - 4x + 1 = (2x-1)^2 \geqq 0$$

it follows that $$x(1-x) \leqq \frac{1}{4}.$$

Lemma 2. *Let* $x \in [0, 1]$ *and* $\delta > 0$ *arbitrarily. If we also denote by* $\Delta_n(x)$ *the set of* k*-values from the range* 0, 1, 2, 3, . . . , n, *for which*

$$\left|\frac{k}{n} - x\right| \geqq \delta, \tag{11}$$

then

$$\sum_{k \in \Delta_n(x)} C_n^k x^k (1-x)^{n-k} \leqq \frac{1}{4n\delta^2} \tag{12}$$

Proof. For $k \in \Delta_n(x)$, it follows from (11) that

$$\frac{(k-nx)^2}{n^2\delta^2} \geqq 1$$

and hence

$$\sum_{k \in \Delta_n(x)} C_n^k x^k (1-x)^{n-k} \leqq \frac{1}{n^2 \delta^2} \sum_{k \in \Delta_n(x)} (k - n x)^2 C_n^k x^k (1-x)^{n-k}$$

If we now extend the summation on the right-hand side to all values in the range $k = 0, 1, 2, 3, \ldots, n$, then the right-hand sum does not decrease, since for $x \in [0, 1]$ none of the newly added summands is negative. Hence the inequality (1) leads immediately to the desired inequality (12).

The gist of this lemma is—in brief—that with extremely large values of n in the sum

$$\sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} \tag{13}$$

only those summands are significant whose index k satisfies the condition

$$\left| \frac{k}{n} - x \right| < \delta ,$$

while the remaining summands contribute but slightly to the total.

Definition. If $f(x)$ is a given function in the segment [0, 1], then every polynomial of the form

$$B_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) C_n^k x^k (1-x)^{n-k}$$

may be denominated a BERNSTEIN *polynomial* of the function $f(x)$.

For large values of n, and if $f(x)$ is continuous, such a polynomial will differ only slightly from $f(x)$. For—as we have already seen—those summands for which $\frac{k}{n}$ is remote from x play hardly any part in the sum (13), and this holds also for the polynomial $B_n(x)$, since all the factors $f\left(\frac{k}{n}\right)$ are of course bounded. In the polynomial $B_n(x)$, accordingly, only those summands are substantially significant for which $\frac{k}{n}$ lies in close proximity to x. But for these summands the factor $f\left(\frac{k}{n}\right)$ differs only slightly from $f(x)$, because of the continuity of $f(x)$. This, however, means that the whole polynomial $B_n(x)$ varies only slightly if we substitute $f(x)$ for $f\left(\frac{k}{n}\right)$ in its summands. In other words: the approximate equation

$$B_n(x) \approx \sum_{k=0}^{n} f(x) C_n^k x^k (1-x)^{n-k}$$

holds good.

From this and equation (1) it immediately follows that

$$B_n(x) \approx f(x).$$

This heuristic consideration is formulated in exact terms in the following theorem:

S. N. Bernstein's Theorem. *If $f(x)$ is continuous in the segment* [0, 1], *then in relation to x*

$$\lim_{n\to\infty} B_n(x) = f(x) \tag{14}$$

holds uniformly in this segment.

Proof. Let M be the greatest value of $|f(x)|$ in [0, 1]. If, furthermore, $\varepsilon > 0$ is arbitrarily assumed, then in consequence of the *uniform* continuity of $f(x)$ in the segment [0, 1], a number $\delta > 0$ can be found such that for

$$|x'' - x'| < \delta$$

the inequality

$$|f(x'') - f(x')| < \frac{\varepsilon}{2}$$

is always true.

Now let x be a value arbitrarily chosen from the segment [0, 1]. From equation (1)

$$f(x) = \sum_{k=0}^{n} f(x) C_n^k x^k (1-x)^{n-k},$$

so that

$$B_n(x) - f(x) = \sum_{k=0}^{n} \left\{ f\left(\frac{k}{n}\right) - f(x) \right\} C_n^k x^k (1-x)^{n-k}. \tag{15}$$

Now let us split the series of values $k = 0, 1, 2, \ldots, n$ into two classes $\Gamma_n(x)$ and $\Delta_n(x)$ by determining:

$$k \in \Gamma_n(x), \quad \text{wenn} \quad \left|\frac{k}{n} - x\right| < \delta,$$

$$k \in \Delta_n(x), \quad \text{wenn} \quad \left|\frac{k}{n} - x\right| \geqq \delta.$$

The sum (15) correspondingly splits also into two sums Σ_Γ and Σ_Δ. In the first

$$\left| f\left(\frac{k}{n}\right) - f(x) \right| < \frac{\varepsilon}{2},$$

whence
$$|\Sigma_\Gamma| \leqq \frac{\varepsilon}{2} \sum_{k \in \Gamma_n(x)} C_n^k x^k (1-x)^{n-k},$$

and as
$$\sum_{k \in \Gamma_n(x)} C_n^k x^k (1-x)^{n-k} \leqq \sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} = 1\ ,$$

it follows that
$$|\Sigma_\Gamma| \leqq \frac{\varepsilon}{2}. \tag{16}$$

In the second sum, if
$$\left| f\left(\frac{k}{n}\right) - f(x) \right| \leqq 2M,$$

then, from (12),
$$|\Sigma_\Delta| \leqq 2M \sum_{k \in \Delta_n(x)} C_n^k x^k (1-x)^{n-k} \leqq \frac{M}{2n\delta^2}.$$

This inequality combined with (16) gives
$$|B_n(x) - f(x)| \leqq \frac{\varepsilon}{2} + \frac{M}{2n\delta^2}.$$

For sufficiently high values $n > N_\varepsilon$
$$\frac{M}{n\delta^2} < \varepsilon \tag{17}$$

and consequently
$$|B_n(x) - f(x)| < \varepsilon.$$

Hence the theorem is proved; for the choice of N_ε is determined only by inequality (17) and is in no way dependent on the value of x selected.

We are now in a position to prove the WEIERSTRASS theorem previously mentioned. In fact the WEIERSTRASS theorem follows directly from the BERNSTEIN theorem, if segment $[a, b]$ coincides with segment $[0, 1]$.[2] Now let segment $[a, b]$ be different from segment $[0, 1]$. Then let us introduce the function
$$\varphi(y) = f[a + y(b-a)]$$

[2] We observe, however, that the BERNSTEIN theorem is more productive than the WEIERSTRASS in this case, as it provides a sequence of well-defined polynomials, while the WEIERSTRASS theorem only establishes the existence of such a sequence of approximations, without stating anything in regard to its construction.

which is defined and continuous in the segment [0, 1]. From what has just been proved, a polynomial

$$Q(y) = \sum_{k=0}^{n} c_k y^k$$

must exist that satisfies the condition

$$\left| f[a + y(b - a)] - \sum_{k=0}^{n} c_k y^k \right| < \varepsilon \tag{18}$$

for all values of $y \in [0, 1]$.

But for every value of $x \in [a, b]$, the fraction $\dfrac{x - a}{b - a}$ lies in the segment [0, 1]; it can therefore be substituted for y in (18). This gives

$$\left| f(x) - \sum_{k=0}^{n} c_k \left(\frac{x - a}{b - a}\right)^k \right| < \varepsilon,$$

which proves that the polynomial

$$P(x) = \sum_{k=0}^{n} c_k \left(\frac{x - a}{b - a}\right)^k$$

approximately represents the function $f(x)$ with the required degree of accuracy.

We can express the WEIERSTRASS theorem in still another form:

A. *Every continuous function $f(x)$ in the segment $[a, b]$ is the limiting function of a uniformly convergent sequence of polynomials in this segment.*

In point of fact, let us take the null-sequence $\varepsilon_n = \dfrac{1}{n}$, then for each such value ε_n a polynomial $P_n(x)$ can be found that satisfies the condition

$$|P_n(x) - f(x)| < \frac{1}{n} \qquad (a \leqq x \leqq b).$$

Hence, clearly

$$P_n(x) \rightrightarrows f(x)$$

for $n \to \infty$.[3]

Finally, we give the WEIERSTRASS theorem still a third form:

B. *Every continuous function in a segment can be expanded into a uniformly converging series of polynomials in that segment.*

Suppose we have found a series of polynomials uniformly converging on $f(x)$, then let

$$Q_1(x) = P_1(x), \quad Q_n(x) = P_n(x) - P_{n-1}(x) \qquad (n > 1).$$

[3] Occasionally, we denote uniform convergence by the symbol $\rightrightarrows$.

Then the partial sums of the series

$$\sum_{n=1}^{\infty} Q_n(x)$$

coincide with the polynomials $P_n(x)$, so that this series itself uniformly converges on $f(x)$.

§ 2. Weierstrass' Second Approximation Theorem

Weierstrass' second theorem states the possibility of approximately representing continuous *periodic* functions by trigonometric polynomials to any required degree of accuracy.

Weierstrass' second theorem. *If $f(x) \in C_{2\pi}$ and $\varepsilon > 0$ be arbitrarily assumed, then there exists a trigonometric polynomial $T(x)$ such that for all real values of x the inequality $|T(x) - f(x)| < \varepsilon$ is satisfied.*

This theorem—like the first—also admits two other formulations (of type A, and B, respectively). A particularly simple proof was furnished for this by De La Vallée-Poussin in 1908 [1]; we follow it here.

Lemma 1. *If $\phi(x) \in C_{2\pi}$, then for all values α the equation*

$$\int_a^{a+2\pi} \varphi(x)\,dx = \int_0^{2\pi} \varphi(x)\,dx$$

holds true.

In fact, we have

$$\int_a^{a+2\pi} \varphi(x)\,dx = \int_a^0 \varphi(x)\,dx + \int_0^{2\pi} \varphi(x)\,dx + \int_{2\pi}^{a+2\pi} \varphi(x)\,dx.$$

If in the last integral on the right we put $x = z + 2\pi$ and consider the equation $\phi(z + 2\pi) = \phi(z)$, then for this last integral we get the value

$$-\int_a^0 \varphi(z)\,dz,$$

from which the lemma follows.

Lemma 2. *The identity* [4]

$$\int_0^{\pi/2} \cos^{2n} t\,dt = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2} \tag{19}$$

holds true.

[4] The symbol $n!!$ denotes the product of all natural numbers not exceeding n and even or uneven according as n is even or uneven, e.g., $8!! = 2 \cdot 4 \cdot 6 \cdot 8$.