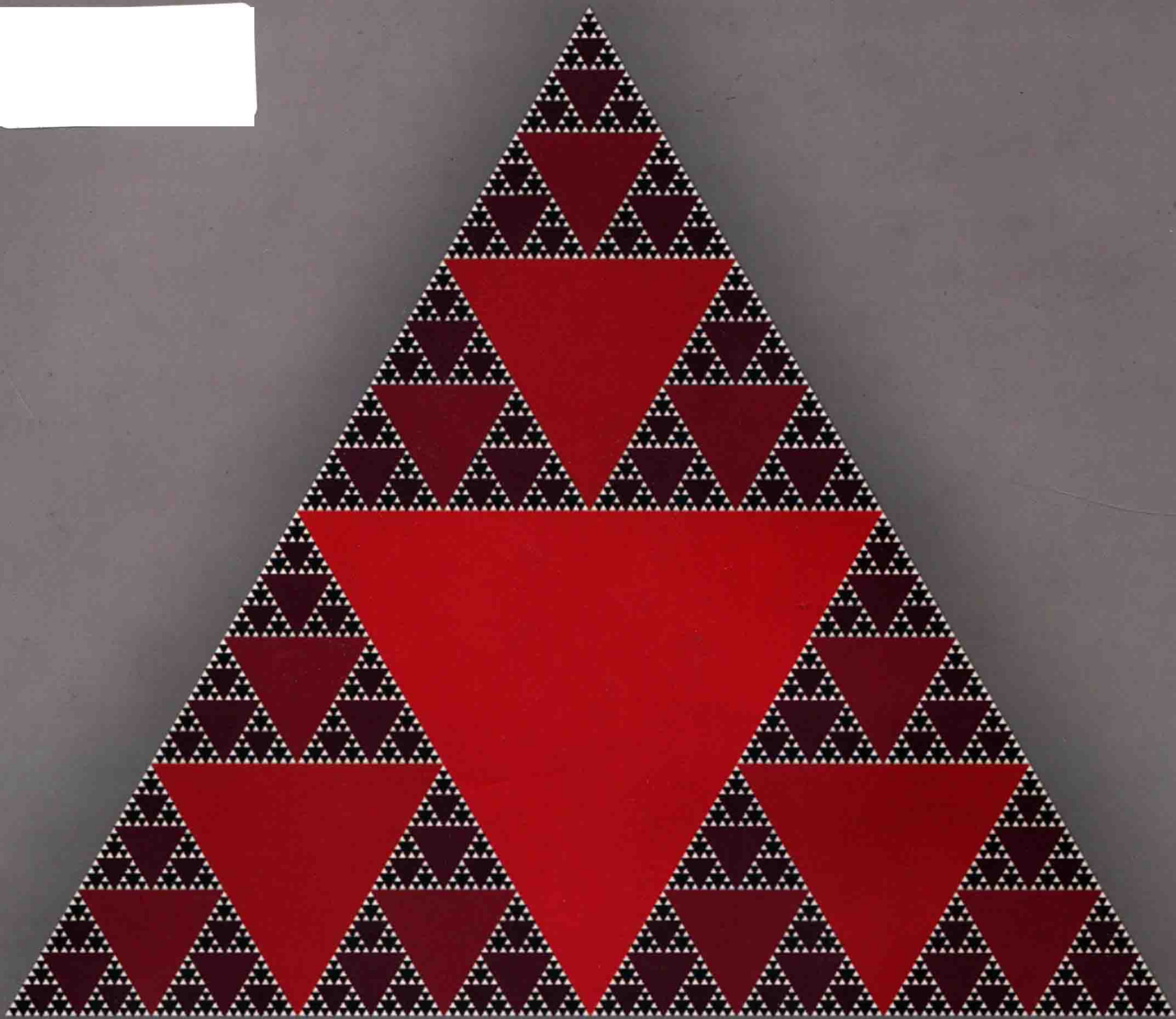


Beautiful Geometry

ELI MAOR AND EUGEN JOST

Beautiful Geometry

ELI MAOR and EUGEN JOST

Princeton University Press
Princeton and Oxford

Published by Princeton University Press, 41 William Street, Princeton, New Jersey 08540

In the United Kingdom: Princeton University Press, 6 Oxford Street, Woodstock, Oxfordshire OX20 1TW

press.princeton.edu

Jacket Illustration: *Sierpinski's Arrowhead* by Eugen Jost

Library of Congress Cataloging-in-Publication Data

Maor, Eli.

Beautiful geometry / Eli Maor and Eugen Jost.

pages cm

Summary: "If you've ever thought that mathematics and art don't mix, this stunning visual history of geometry will change your mind. As much a work of art as a book about mathematics, Beautiful Geometry presents more than sixty exquisite color plates illustrating a wide range of geometric patterns and theorems, accompanied by brief accounts of the fascinating history and people behind each. With artwork by Swiss artist Eugen Jost and text by acclaimed math historian Eli Maor, this unique celebration of geometry covers numerous subjects, from straightedge-and-compass constructions to intriguing configurations involving infinity. The result is a delightful and informative illustrated tour through the 2,500-year-old history of one of the most important and beautiful branches of mathematics"— Provided by publisher.

Includes index.

ISBN-13: 978-0-691-15099-4 (cloth : acid-free paper)

ISBN-10: 0-691-15099-0 (cloth : acid-free paper) 1. Geometry—History. 2. Geometry—History—Pictorial works. 3. Geometry in art. I. Jost, Eugen, 1950– II. Title.

QA447.M37 2014

516—dc23

2013033506

British Library Cataloging-in-Publication Data is available

This book has been composed in Baskerville 10 Pro

Printed on acid-free paper. ∞

Printed in Canada

1 3 5 7 9 10 8 6 4 2

To Dalia, my dear wife of fifty years

May you enjoy many more years of

good health, happiness, and Naches from your family.

—Eli

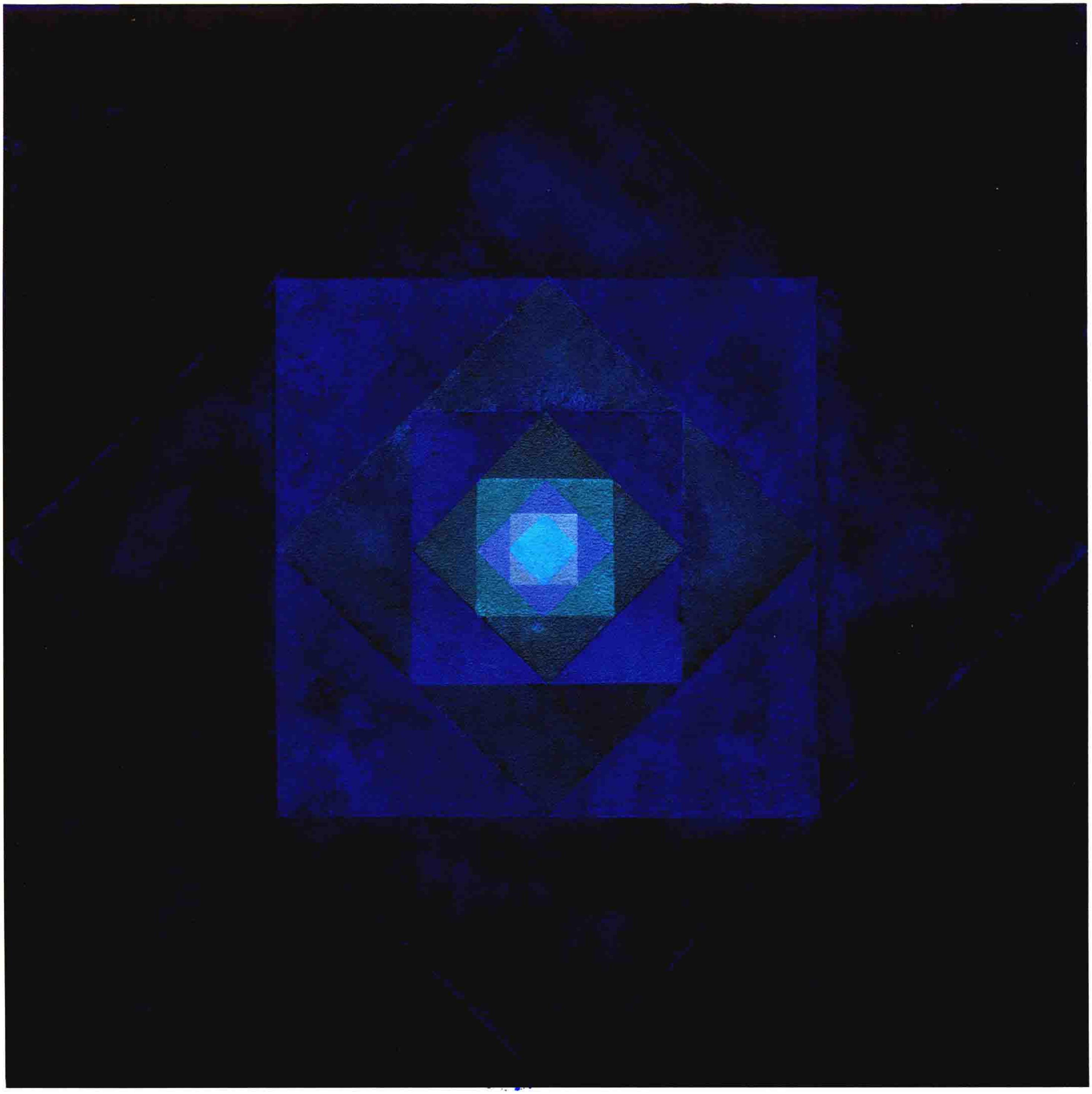
To my dear Kathrin and to my whole family

Two are better than one; because they have a good reward for their labor.

For if they fall, the one will lift up his fellow (Ecclesiastes 4:9-10).

—Eugen

Beautiful Geometry



Frontispiece: *Infinity*

Prefaces

ART THROUGH MATHEMATICAL EYES

ELI MAOR

No doubt many people would agree that art and mathematics don't mix. How could they? Art, after all, is supposed to express feelings, emotions, and impressions—a subjective image of the world as the artist sees it. Mathematics is the exact opposite—cold, rational, and emotionless. Yet this perception can be wrong. In the Renaissance, mathematics and art not only were practiced together, they were regarded as complementary aspects of the human mind. Indeed, the great masters of the Renaissance, among them Leonardo da Vinci, Michelangelo, and Albrecht Dürer, considered themselves as architects, engineers, and mathematicians as much as artists.

If I had to name just one trait shared by mathematics and art, I would choose their common search for pattern, for recurrence and order. A mathematician sees the expression $a^2 + b^2$ and immediately thinks of the Pythagorean theorem, with its image of a right triangle surrounded by squares built on the three sides. Yet this expression is not confined to geometry alone; it appears in nearly every branch of mathematics, from number theory and algebra to calculus and analysis; it becomes a pattern, a paradigm. Similarly, when an artist looks at a wallpaper design, the recurrence of a basic motif, seemingly repeating itself to infinity, becomes etched in his or her mind as a pattern. *The search for pattern* is indeed the common thread that ties mathematics to art.

The present book has its origin in May 2009, when my good friend Reny Montandon arranged for me to give a talk to the upper mathematics class of the Alte Kantonsschule (Old Cantonal High School) of Aarau, Switzerland. This school has a historic claim to fame: it was here that a 16-year-old Albert Einstein spent two of his happiest years, enrolling there at his own initiative to escape the authoritarian educational system he so much loathed at home. The school still occupies the same building that Einstein knew, although a modern wing has been added next to it. My wife and I were received with great honors, and at lunchtime I was fortunate to meet Eugen Jost.

I had already been acquainted with Eugen's exquisite mathematical artwork through our mutual friend Reny, but to meet him in person gave me special pleasure, and we instantly bonded. Our encounter was the spark that led us to collaborate on the present book. To our deep regret, Reny Montandon passed away shortly before the completion of our book; just one day before his death, Eugen spoke to him over the phone and told him about the progress we were making, which greatly pleased him. Sadly he will not be able to see it come to fruition.

Our book is meant to be enjoyed, pure and simple. Each topic—a theorem, a sequence of numbers, or an intriguing geometric pattern—is explained in words and accompanied by one or more color plates of Eugen's artwork. Most topics are taken from geometry; a few deal with numbers and numerical progressions. The chapters are largely independent of one another, so the reader can choose what he or she likes without affecting the continuity of reading. As a rule we followed a chronological order, but occasionally we grouped together subjects that are related to one another mathematically. I tried to keep the technical details to a minimum, deferring some proofs to the appendix and referring others to external sources (when referring to books already listed in the bibliography, only the author's name and the book's title are given). Thus the book can serve as an informal—and most certainly not complete—survey of the history of geometry.

Our aim is to reach a broad audience of high school and college students, mathematics and science teachers, university instructors, and laypersons who are not afraid of an occasional formula or equation. With this in mind, we limited the level of mathematics to elementary algebra and geometry ("elementary" in the sense that no calculus is used). We hope that our book will inspire the reader to appreciate the beauty and aesthetic appeal of mathematics and of geometry in particular.

Many people helped us in making this book a reality, but special thanks go to Vickie Kearn, my trusted editor at Princeton University Press, whose continuous enthusiasm and support has encouraged us throughout the project; to the editorial and technical staff at Princeton University Press for their efforts to ensure that the book meets the highest aesthetic and artistic standards; to my son Dror for his technical help in typing the script of plate 26 in Hebrew; and, last but not least, to my dear wife Dalia for her steady encouragement, constructive critique, and meticulous proofreading of the manuscript.

PLAYING WITH PATTERNS, NUMBERS, AND FORMS

EUGEN JOST

My artistic life revolves around patterns, numbers, and forms. I love to play with them, interpret them, and metamorphose them in endless variations. My motto is the Pythagorean motto: *Alles ist Zahl* ("All is Number"); it was the title of an earlier project I worked on with my friends Peter Baptist and Carsten Miller in 2008. *Beautiful Geometry* draws on some of the ideas expressed in that earlier work, but its conception is somewhat different. We attempt here to depict a wide selection of geometric theorems in an artistic way while remaining faithful to their mathematical message.

While working on the present book, my mind was often with Euclid: A point is that which has no part; a line is a breadthless length. Notwithstanding that claim, Archimedes drew his broad-lined circles with his finger in the sand of Syracuse. Nowadays it is much easier to meet Euclid's demands: with a few clicks of the mouse you can reduce the width of a line to nothing—in the end there remains only a nonexisting path. It was somewhat awe inspiring to go through the constructions that were invented—or should I say discovered?—by the Greeks more than two thousand years ago.

For me, playing with numbers and patterns always has top priority. That's why I like to call my pictures playgrounds, following a statement by the Swiss Artist Max Bill: "perhaps the goal of concrete art is to develop objects for mental use, just like people created objects for material use." Some illustrations in our book can be looked at in this sense. The onlooker is invited to play: to find out which rules a picture is built on and how the many metamorphoses work, to invent his or her own pictures. In some chapters the relation between text and picture is loose; in others, however, artistic claim stood behind the goal to enlighten Eli's text. Most illustrations were created on the keypad of my computer, but others are acrylics on canvas. Working with Eli was a lot of fun. He is one of those mathematicians that teach you: Mathematics did not fall from heaven; it was invented and found by humans; it is full of stories; it is philosophy, history and culture. I hope the reader will agree.

Beautiful Geometry

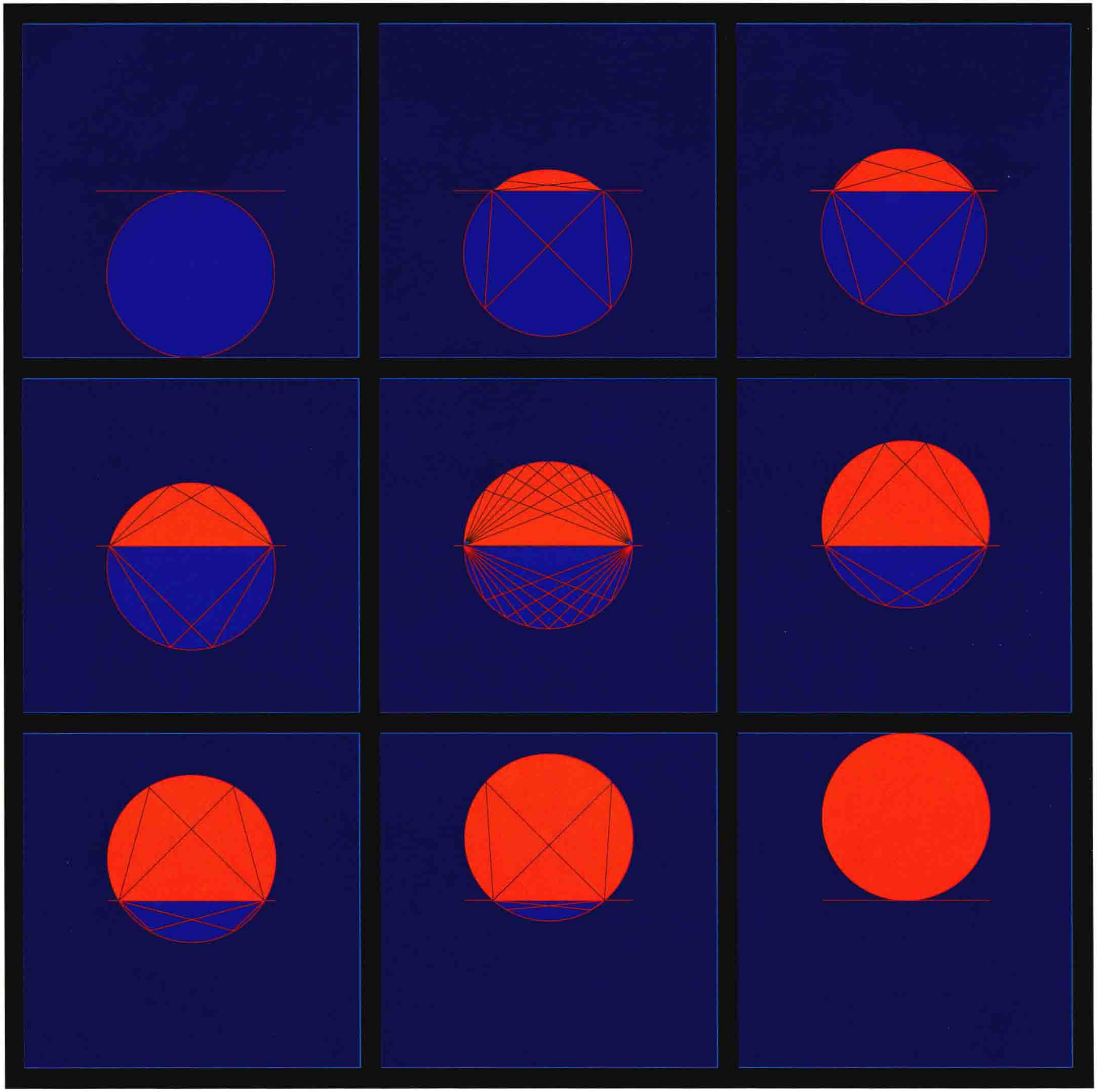


Plate 1. *Sunrise over Miletus*

Contents

1

Thales of Miletus

Thales (ca. 624–546 BCE) was the first of the long line of mathematicians of ancient Greece that would continue for nearly a thousand years. As with most of the early Greek sages, we know very little about his life; what we do know was written several centuries after he died, making it difficult to distinguish fact from fiction. He was born in the town of Miletus, on the west coast of Asia Minor (modern Turkey). At a young age he toured the countries of the Eastern Mediterranean, spending several years in Egypt and absorbing all that their priests could teach him.

While in Egypt, Thales calculated the height of the Great Cheops pyramid, a feat that must have left a deep impression on the locals. He did this by planting a staff into the ground and comparing the length of its shadow to that cast by the pyramid. Thales knew that the pyramid, the staff, and their shadows form two similar right triangles. Let us denote by H and h the heights of the pyramid and the staff, respectively, and by S and s the lengths of their shadows (see figure 1.1). We then have the simple equation $H/S = h/s$, allowing Thales to find the value of H from the known values of S, s, and h. This feat so impressed Thales's fellow citizens back home that they recognized him as one of the Seven Wise Men of Greece.

Mathematics was already quite advanced during Thales's time, but it was entirely a practical science, aimed at devising formulas for solving a host of financial, commercial, and engineering problems. Thales was the first to ask not only *how* a particular problem can be solved, but *why*. Not willing to accept facts at face value, he attempted to prove them from fundamental principles. For example, he is credited with demonstrating that the two base angles of an isosceles triangle are equal, as are the two vertical angles formed by a pair of intersecting lines. He also showed that the diameter of a circle cuts it into two equal parts, perhaps by folding over the two halves and observing that they exactly overlapped. His proofs may not stand up to modern standards, but they were a first step toward the kind of deductive mathematics in which the Greeks would excel.

Thales's most famous discovery, still named after him, says that from any point on the circumference of a circle, the diameter always subtends a right angle. This was perhaps the first known *invariance* theorem—the fact that in a geometric configuration,

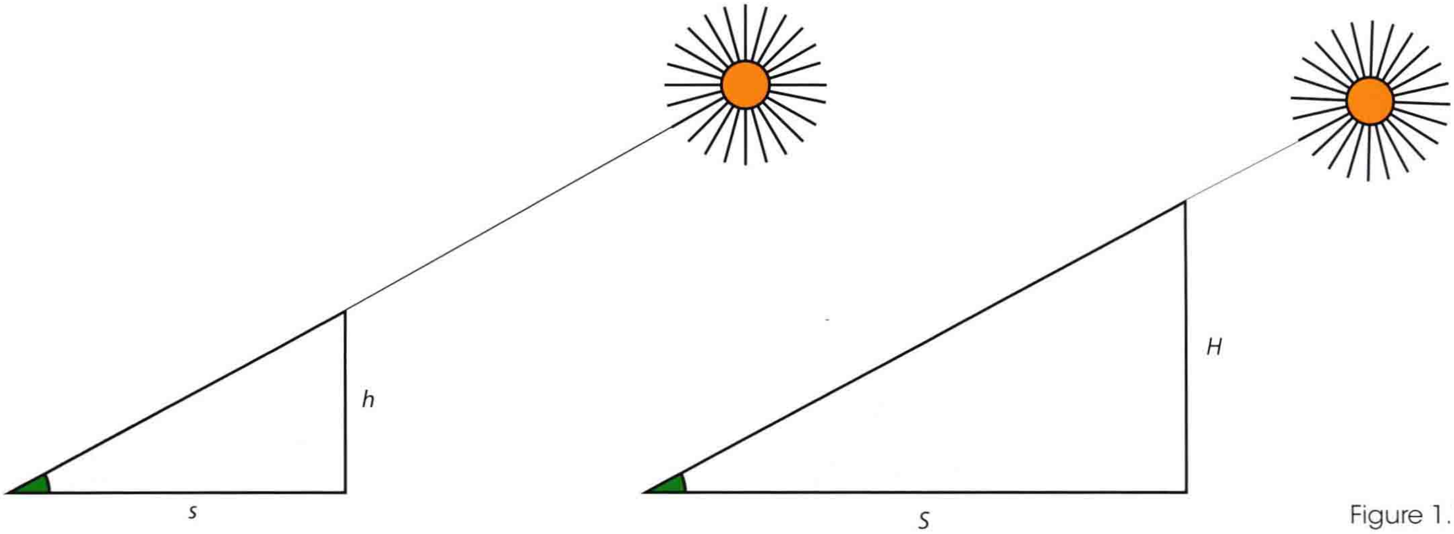


Figure 1.1

some quantities remain the same even as others are changing. Many more invariance theorems would be discovered in the centuries after Thales; we will meet some of them in the following chapters.

Thales's theorem can actually be generalized to any chord, not just the diameter. Such a chord divides the circle into two unequal arcs. Any point lying on the larger of these arcs subtends the chord at a constant angle $\alpha < 90^\circ$; any point on the smaller arc subtends it at an angle $\beta = 180^\circ - \alpha > 90^\circ$.[1] Plate 1, *Sunrise over Miletus*, shows this in vivid color.

NOTE:

1. The converse of Thales's theorem is also true: the locus of all points from which a given line segment subtends a constant angle is an arc of a circle having the line segment as chord. In particular, if the angle is 90°, the locus is a full circle with the chord as diameter.