S.Sternberg

Group theory and physics

群论和物理学

CAMBRIDGE
世界图书出版公司

GROUP THEORY AND PHYSICS

S. STERNBERG

GEORGE PUTNAM PROFESSOR
OF PURE AND APPLIED MATHEMATICS,
HARVARD UNIVERSITY
AND PERMANENT SACKLER FELLOW,
UNIVERSITY OF TEL AVIV.

CAMBRIDGE
UNIVERSITY PRESS

世界图书出版公司

书　　名：Group Theory and Physics
作　　者：S. Sternberg
中 译 名：群论和物理学
出 版 者：世界图书出版公司北京公司
印 刷 者：北京世图印刷厂
发　　行：世界图书出版公司北京公司 (北京朝内大街 137 号　100010)
开　　本：1/24　711×1245　印　张：18.5
出版年代：2000 年 4 月
书　　号：ISBN7-5062-4965-0/O・316
版权登记：图字 01-2000-0857
定　　价：74.00 元

世界图书出版公司北京公司已获得 Cambridge University Press 授权在中国大陆独家重印发行。

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge CB2 1RP, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, United Kingdom
40 West 20th Street, New York, NY 10011-4211, USA
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

© Cambridge University Press 1994

This book is in copyright. Subject to statutory exception
and to the provisions of relevant collective licensing agreements,
no reproduction of any part may take place without
the written permission of Cambridge University Press.

First published 1994
Reprinted 1995
First paperback edition 1995
Reprinted 1997

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this book is available from the British Library

ISBN 0 521 24870 1 hardback
ISBN 0 521 55885 9 paperback

This edition of Group Theory of Physics by Shlomo Sternberg is published by arrangement with the Syndicate of the Press of University of Cambridge, Cambridge, England.

Licensed edition for sale in the People's Republic of China only. Not for export elsewhere.

PREFACE

Group theory is one of the great achievements of 19th century mathematics. It emerged as a unifying idea drawing on four different sources: number theory, the theory of equations, geometry, and crystallography. The early motivation from number theory stemmed from the work of Euler, Legendre and Gauss on power residues. In the theory of equations, the study of various permutation groups became increasingly important through the work of Lagrange, Ruffini, Gauss, Abel, Cauchy, and especially Galois. The discovery of new types of geometries – including non-Euclidean, affine, projective etc. – led, eventually, to the famous Erlangen program of Klein, which proposed that the true study of any geometry lies in an analysis of its group of motions. In crystallography, the possible symmetries of the internal structure of a crystal were enumerated long before there was any possibility of its physical determination (by X-ray analysis).

The definition of an abstract group was proposed by Cayley in two remarkable papers in 1854, reflecting perhaps some influence of Boole (for abstract formulation) and Hamilton's quaternions (for the existence of algebras with noncommutative multiplication). This definition was not immediately appreciated by the mathematical community. In 1870, Kronecker (independently of Cayley) introduced the axioms for an abstract commutative group. When Cayley reiterated his definition in 1878, the reception was much warmer. In the period from 1870 to 1900, enormous progress was made in group theory. For example, the idea of a continuous group was introduced and studied by Lie; this led to a wealth of applications to geometry and differential equations, culminating in the classification by Killing and Cartan of the simple finite dimensional Lie groups. The theory of finite groups was greatly advanced through the work of Jordan, Hölder and Burnside.

The theory of group representations was created by Frobenius, Schur and Burnside in the last decade of the 19th century, although some of the ideas were anticipated in Jordan's monumental *Traité des substitutions* of 1870. Their theory for finite groups was extended to compact groups and brought into fruitful contact with Lie theory in a series of fundamental papers by Hermann Weyl in the 1920s. Almost all the key mathematical ideas presented in this book were developed during this period; thus, some 70 to 120 years ago. (The one principal exception is our description of Wigner's seminal paper, extending Frobenius's method to obtain the representations of the Poincaré group. This paper appeared around 50 years ago.) Of course, there has been

此为试读,需要完整PDF请访问: www.ertongbook.com

huge progress in the last half century. I have tried to present this classical material from a geometric viewpoint, which will, hopefully, help the reader to enter the realm of the more recent advances.

It is more difficult to trace the early sources of the applications of group theory to physics. Symmetry considerations entered into the solutions of physical problems at the very beginning of mathematical physics. Mathematical crystallography, a major success of 19th century physics, is essentially group theoretical, but it had developed before the abstract language of group theory had been accepted. We explain some of the more elementary ideas of this subject in Chapter 1, and go into somewhat more detail in Appendix A. The spirit of Klein's Erlangen program pervades Poincaré's *La Science et l'Hypothese*, and other philosophical writings, and through them influenced the development of special relativity. The culmination of this group theoretical approach to relativity is Wigner's paper mentioned above, where the physical characteristics, mass and spin, arise as parameters in the description of irreducible representations. One of the goals of our method of presentation is to reach this central result.

The explicit recognition of the importance of group representation theory in physics started very soon after the discovery of quantum mechanics, with the path-breaking work of Weyl, Wigner, and others. In fact, Weyl's classic book of 1928, *Gruppentheorie und Quantenmechanik*, makes instructive and inspiring reading even today. (In his book, Weyl adopts the pedagogic strategy of segregating the mathematics and the physics into separate chapters. There is much to be said for this strategy, especially from the point of view of logical coherence. But it had the unintended effect that physicists and mathematicians would read alternate chapters. I have taken the risk of going to the opposite extreme here, trying to use the physics to motivate the mathematics and vice versa, mixing the two.) The uses of group theory in quantum mechanics extended from chemistry and spectroscopy in the 1920s and 1930s, to nuclear and particle physics in the 1930s and 1940s, and then to high energy physics and the discovery of the theory of colored quarks in the 1960s and 1970s. It is this story of the interweaving of mathematics and physics that I try to tell in this book.

It should not be supposed that there was a warm reception in the physics community to the introduction of group theoretical methods. In fact, the contrary was true. To get a feeling for a typical early reaction, let me quote at length from the autobiography of John Slater, who was a leading American physicist and head of the MIT Physics Department for many years. The following quotes are taken from pages 60–2 of his autobiography:

> It was at this point that Wigner, Hund, Heitler, and Weyl entered the picture with their "Gruppenpest": the pest of the group theory.... The authors of the "Gruppenpest" wrote papers which were incomprehensible to those like me who had not studied group theory, in which they applied these theoretical results to the study of the many electron problem. The practical consequences appeared to be negligible, but everyone felt that to be in the mainstream one had to learn about it. Yet there were no good texts from which one could learn group theory. It was a frustrating experience, worthy of the name of a pest.

> I had what I can only describe as a feeling of outrage at the turn which the subject had taken....
>
> As soon as this [Slater's] paper became known, it was obvious that a great many other physicists were as disgusted as I had been with the group-theoretical approach to the problem. As I heard later, there were remarks made such as "Slater has slain the 'Gruppenpest'". I believe that no other piece of work I have done was so universally popular.

Outrage, disgust, the characterization of group theory as a plague or as a dragon to be slain – this is not an atypical physicist's reaction in the 1930s–50s to the use of group theory in physics. It is, however, amazing to consider that this autobiography was published in 1975, after the major triumphs of group theory in elementary particle physics.

When I was a student in the early 1950s, the basic facts of abstract group theory were part of the algebra course, but the theory of group representations was not included in the standard mathematics curriculum. My introduction to representation theory and its physical applications was at the hands of Prof. George W. Mackey. He gave a wonderful and justly famous course of lectures at the University of Chicago in the summer of 1955, where I visited as a special summer student. From the time that I joined the Harvard faculty in 1959, George has given me access to his voluminous handwritten notes on mathematical physics, and has, on occasion, written me long letters explaining various points. Much of his influence can be felt in the first half of this book.

In 1962, I was invited by Prof. Yuval Ne'eman to give a series of lectures on the topology of Lie groups at his seminar, then held at Nahal Soreq. This was after the prediction of the existence of the Ω^- particle (by Gell-Mann and by Ne'eman on the basis of $SU(3)$ symmetry in 1961), but before its momentous discovery at Brookhaven National Laboratory in 1964. This series of lectures developed into a lifelong collaboration. Much of my own work in physics has been in collaboration with Prof. Ne'eman, or an outgrowth of the seminars we have held together over the past 32 years.

Let me now describe the contents. The key idea in Chapter 1 is an action of a group on a set, with the concomitant notions of fixed point sets and stabilizer subgroups. We use these notions to clarify the notion of form and habit in a crystal, and to classify the finite subgroups of $O(3)$. Along the way we show that $Sl(2, \mathbb{C})$ is the double cover of the connected component of the Lorentz group, and hence that $SU(2)$ is a double cover of the three-dimensional rotation group, facts that are central to the understanding of the concept of spin. We conclude with a discussion of icosahedral symmetry in conjunction with the newly discovered carbon molecules, the buckyballs.

Chapter 2 presents the basic facts in the representation theory of finite groups. The central unifying theme is that of character formulas as fixed point formulas, both in this chapter and the next. Of course, we present these formulas in the purely finite context. But they represent the finite prototypes of the more powerful fixed point formulas in modern analysis, such as the Atiyah–Bott theorem. A partial

transition to these formulas is given in Appendix G using differential geometric methods and generalized functions. At the end of the chapter we make a first pass at the representation theory of the symmetric groups. I return to this topic in Chapter 5 and in Appendix C.

Chapter 3 discusses induced representations from the point of view of vector bundles. The motivating example is the study of the vibrational spectrum of a molecule, both in classical and quantum mechanics. The main physical idea is the use of Schur's lemma to determine the number of possible vibrational modes and also to derive the quantum mechanical selection rules that determine which transitions are forbidden. In this latter connection, one needs to use tensor products and what are known in the physics literature as 'tensor operators'. I give the Frobenius theory of the representations of a semidirect product, and describe Wigner's use of this method to obtain the irreducible representations of the symmetry group of special relativity. The chapter includes a careful mathematical discussion of the question of the discrete symmetries of space time such as parity and time reversal. The chapter concludes with the Mackey theorems on induced representations and Mackey's approach to exchange forces.

Chapter 4 makes the transition from finite to compact groups. The Peter–Weyl theorem is stated, but its proof is deferred to Appendix E. The irreducible representations of $SU(2)$ and $SO(3)$ are derived, with the concomitant theory of spherical harmonics. Applications include a discussion of the hydrogen spectrum and the role of the representation theory of the rotation group in the periodic table and the magic numbers of nuclear physics. I discuss the role of the Clebsch–Gordan coefficients in isospin, in particular in pion–nucleon scattering experiments, and show how the Klein–Gordan equation, the Dirac equation, Weyl's neutrino equation and Maxwell's equation are related to the appropriate irreducible representations of the Poincaré group in Wigner's list. A brief introduction to Lie algebra methods is included.

Chapter 5 is devoted to the Schur–Weyl duality between representations of the symmetric groups and the general linear groups. The representations of the special unitary groups are derived from this duality in the standard fashion. The results are applied to the study of quarks. A typical application is the derivation of the nucleon magnetic moments from the quark theory. The physics in this chapter represents discoveries up to the early 1970s. I do not include a discussion of electroweak unification, but end, somewhat out of context, with a discussion of the differential geometry of the Higgs mechanism. I have not included any of the material on grand unified or supersymmetric models. My feeling is that no one of these models has won the day, and that the fundamental problems, such as confinement, the mass spectrum, divergences, the source of the Higgs field, etc., must be regarded as open questions. In the meantime, the attention of much of the theoretical physics community has turned elsewhere.

There are seven appendices. Appendix A takes the study of mathematical crystallography a bit further than does the treatment in Chapter 1. It does not go through the detailed classification of the crystallographic groups, but does give a description of how this classification proceeds. In particular, it gives a precise definition of the Bravais lattices and their classification. My feeling is that this material is of general

cultural importance (and is, of course, central to solid state physics) but too technical to be included in the introductory first chapter.

Appendix B provides the necessary background on tensor products, as this material is not always included in the standard linear algebra course.

Appendix C provides proofs of some of the more technical aspects of the representation theory of the symmetric group. I chose to follow the beautiful 1977 paper of James. This method illustrates the power of the Gel'fand approach to integral geometry. Alternative approaches to this theory, such as via Hopf algebras or combinatorics, have their own individual merits, and are available elsewhere.

Appendix D gives the proof (following Bargmann) of Wigner's theorem on the symmetries of quantum logic. This theorem lies at the heart of the application of group theory to quantum mechanics. It is the quantum mechanical version of the fundamental theorem of projective geometry. As it is not usually included in the standard texts, I thought it important to include it here.

Appendix E gives the proofs of the basic facts about the representations of compact groups. The treatment is concise and standard, and practically no hard theorems in functional analysis are used. Nevertheless, I felt that it would be too much of a distraction from the main storyline to include this material in the main text.

Appendix F includes no mathematics at all. It is devoted to a history of 19th century spectroscopy. Many quantum mechanics texts start with a little of the prehistory of the subject, usually beginning with the Bohr atom. But it took a century of research to reach Bohr's epoch making paper, and during most of that period the existence of atoms was in dispute, not to say the existence of subatomic constituents. My feeling is that we are in a similar state today with regards to quarks and their possible constituent components. So a look back at how the science of spectroscopy actually progressed might be a source of comfort and amusement during our present period of groping towards an understanding of the deeper components of matter. My guides to the original literature were early Encyclopedia Britannica articles, all written by key players (that is how it was in those days) and the excellent unpublished Ph.D. thesis by Clifford Lawrence Meier entitled 'The role of spectroscopy in the acceptance of an internally structured atom 1860–1920', submitted to the University of Wisconsin in 1964. Of course, I bear the responsibility for the judgment calls in the shaping of the story.

Appendix G is taken from my joint book *Geometric Asymptotics*, written with Victor Guillemin. I try to give a taste of how the fixed point theorems given in the text in the finitistic setting can be formulated and proved in the framework of differential topology.

A word about prerequisites. I have tried to make the demands on the mathematical background of the reader as modest as possible. A course in multivariate calculus and linear algebra, together with an elementary physics course, should suffice. Especially if the reader will forgive my occasional lapses into more advanced material.

תושלב"ע

CONTENTS

此为试读,需要完整PDF请访问: www.ertongbook.com

The identity element is the identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the inverse of each element is its matrix inverse; for example,

$$a^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = c.$$

(c) Let C_4 denote the group of rotational symmetries of the square, as follows:

e = identity (rotation through 0)
a = counterclockwise rotation through $\pi/2$
b = counterclockwise rotation through π
c = counterclockwise rotation through $3\pi/2$ (clockwise rotation through $\pi/2$).

Now the group operation is composition of transformations. Clearly the 'multiplication table' is the same as in the preceding two examples; we have considered three different realizations of the same abstract group, the so-called 'cyclic group of four elements'. It is a simple example of a *finite* group.

Example 2

We turn now to an example of a group which has an infinite number of elements. Let $SL(2, \mathbb{C})$ denote the set of 2×2 matrices of determinant 1 with complex entries. Thus, an element A of $SL(2, \mathbb{C})$ is given as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a, b, c and d are complex numbers satisfying

$$ad - bc = 1.$$

Multiplication is the ordinary multiplication of matrices. Since the determinant of the product of two matrices is the product of their determinants, we see that if A and B are elements of $SL(2, \mathbb{C})$, then so is their product AB. If A is an element of $SL(2, \mathbb{C})$, so that $\det A = 1$, then A is invertible and $\det A^{-1} = 1$, so that A^{-1} exists and lies in $SL(2, \mathbb{C})$. The identity element of the group is the identity matrix, i.e.

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The associative law holds for matrix multiplication and thus $SL(2, \mathbb{C})$ is indeed a group. Notice that the commutative law does not hold in general for this group.

More generally, we can consider $n \times n$ matrices with either real or complex entries. The collection of real invertible $n \times n$ matrices is denoted by $GL(n, \mathbb{R})$. (Notice that here the condition of invertibility has to be added as a supplemental hypothesis. Not all

2×2 or $n \times n$ matrices are invertible, but those that are invertible form a group.) The group $GL(n, \mathbb{R})$ is called the *real general linear* group in n variables. We can also consider the group $SL(n, \mathbb{R})$ consisting of the $n \times n$ real matrices of determinant 1. It is called the *real special linear* group in n variables. Similarly, we can consider the group $GL(n, \mathbb{C})$ of all invertible complex $n \times n$ matrices or the group $SL(n, \mathbb{C})$ of all $n \times n$ complex matrices of determinant 1.

Example 3

As a third example of a group we can consider the group, $O(3)$, of all orthogonal transformations in Euclidean three-dimensional space. This is the group of all linear transformations of three-dimensional space which preserve the Euclidean distance; that is, those transformations, A, which satisfy

$$\|A\mathbf{v}\| = \|\mathbf{v}\|$$

for all vectors $\mathbf{v}$ in ordinary three-dimensional space. If we choose an orthonormal basis for three-dimensional space so that every A becomes identified with a matrix, then A is an orthogonal transformation if and only if

$$AA^t = e,$$

where e denotes the identity matrix in three dimensions. Notice that this equation is the same as $A^t = A^{-1}$. We see immediately that the product of any two orthogonal transformations is again orthogonal and that the inverse of any orthogonal transformation exists and is orthogonal. Thus, the collection of all orthogonal transformations does indeed form a group. Since $\det A = \det A^t$, it follows from $AA^t = e$ that $(\det A)^2 = 1$. Thus, for any orthogonal transformation A we have $\det A = \pm 1$. The collection of those matrices which are orthogonal, and which satisfy the further condition that $\det A = +1$, forms a subcollection of $O(3)$, which in itself is a group and which we will denote by $SO(3)$. We say that $SO(3)$ is a *subgroup* of $O(3)$. $SO(3)$ is called the special orthogonal group in three variables. (Similarly, $SL(n, \mathbb{C})$ is a subgroup of $GL(n, \mathbb{C})$, and $SL(n, \mathbb{R})$ is a subgroup of $GL(n, \mathbb{R})$.) More generally, if we put the standard Euclidean scalar product on the n-dimensional space $\mathbb{R}^n$, we can consider the orthogonal group $O(n)$ of all orthogonal $n \times n$ matrices and the corresponding subgroup $SO(n)$ of those orthogonal matrices with determinant 1.

Example 4

Let $\mathbb{C}^n$ denote the n-dimensional complex vector space of all complex n-tuples with its standard Hermitian scalar product, so that

$$(\mathbf{z}, \mathbf{w}) = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n,$$

where

$$\mathbf{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}.$$

Table 1.

	1	−1
1	1	−1
−1	−1	1

Table 2.

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

A complex matrix A is *unitary* if

$$(Az, Aw) = (z, w)$$

for all z and w in $\mathbb{C}^n$. If we denote $\overline{A^t}$ (the complex conjugate transpose of A) by A^*, we may say that A is unitary only if $AA^* = e$. The product of two unitary matrices is unitary, and the inverse of a unitary matrix is unitary; so the collection of unitary $n \times n$ matrices forms a group which we denote by $U(n)$. Since $\det A^* = \overline{\det A}$, we see that $|\det A| = 1$ for A in $U(n)$. The subgroup of $U(n)$ consisting of those matrices which in addition satisfy $\det A = 1$ is denoted by $SU(n)$.

Thus, for example, the group $SU(2)$ consists of all 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad \text{where} \quad |a|^2 + |b|^2 = 1.$$

Example 5

We can generalize Examples 1(*a*), (*b*) and (*c*) by replacing the number 4 by any positive integer. For instance, we can consider the group C_2 consisting of two elements with the 'multiplication table' as in Table 1, which is isomorphic to the additive group of the integers modulo 2. Similarly, we can think of the three-element group, C_3, with elements $1, \omega, \omega^2$ where $\omega = \exp 2\pi i/3$ which obey the 'multiplication table' shown in Table 2.

The group C_3 can be thought of as the additive group of the integers modulo 3, or as the group of all rotations in the plane which preserve an equilateral triangle centered at the origin. Thus, ω represents rotation through $2\pi/3 = 120°$.

• We have already considered the group C_4 of all rotations preserving a square. It

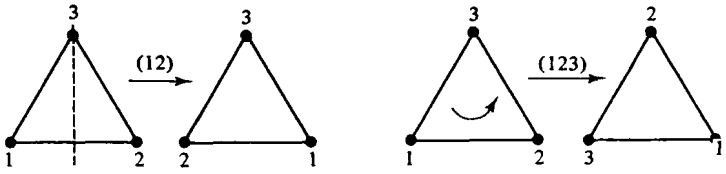

Fig. 1.1

contains four elements, consisting of the identity, rotation through $\pi/2$, rotation through π, and rotation through $3\pi/2$. We can now recognize that C_4 is a subgroup of $SO(2)$, the group of all rotations in the plane. More generally, we can consider C_n as the group of all rotations which preserve a regular polygon with n sides. It will consist of the identity and all rotations through angles of the form $2\pi k/n$.

Example 6

Let us go back to the equilateral triangle. We can consider the group of *all* symmetries of the triangle, not only the rotations. That is, we can allow reflection about perpendicular bisectors as well. This group has six elements; we will denote it by S_3. Notice that we can find some element in S_3 which has the effect of making any desired permutation of the vertices of the triangle. Let us denote the vertices of the triangle by 1, 2 and 3 (Fig. 1.1). Suppose, for example, that (12) denotes the permutation which interchanges the vertices 1 and 2 but leaves a third vertex, 3, fixed. This permutation can be achieved by a reflection about the perpendicular bisector of the edge joining 1 to 2.

Similarly, let (123) denote the permutation that sends 1 into 2, 2 into 3, and 3 into 1. This can be achieved by rotating the triangle through 120°. The permutation (132), which sends 1 into 3, 3 into 2, and 2 into 1, is achieved by rotating the triangle through 240°. From this we see that the group of symmetries of an equilateral triangle is the same as the group of all permutations on three symbols.

Suppose we consider four symbols 1, 2, 3, 4, instead of three. Let s be a one-to-one map of this four-element set onto itself. Thus, s is a permutation of this four-element set. There are four possibilities for $s(1)$: it can be any of the numbers 1, 2, 3, 4. Once we know what $s(1)$ is, then there are three remaining possibilities for $s(2)$, then two remaining possibilities for $s(3)$. Finally, $s(4)$ will be completely determined by being the last remaining number. Thus, there are $4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$ permutations on four letters. The group S_4 is the group of all these permutations. Similarly, we define the group S_n to be the group of all permutations; that is, all one-to-one transformations on a set with n elements.

Example 7

As a final example, we consider the group of all symmetries of the square, denoted by D_4. D_4 contains eight elements: four rotations, together with four reflections – the reflections about the two diagonals, and the reflections about the two perpendicular bisectors (see Fig. 1.2).

Each element of D_4 permutes the vertices 1, 2, 3, 4 of the square. Thus, we may regard

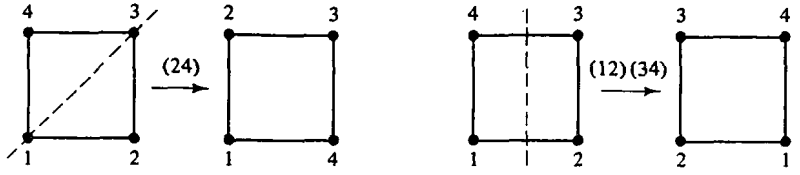

Fig. 1.2

D_4 as a subgroup of S_4, but not every element of S_4 (which has 24 elements all together) lies in D_4, which has only eight elements. Similarly, the group D_n, the group of symmetries of the regular polygon with n sides, is a subgroup of the group S_n of permutations of n symbols. The reader should check that D_n contains $2n$ elements.

1.2 Homomorphisms: the relation between $SL(2,\mathbb{C})$ and the Lorentz group

Let G_1 and G_2 be groups. Let ϕ be a map from G_1 to G_2. We say that ϕ is a *homomorphism* if

$$\phi(ab) = \phi(a)\phi(b) \quad \text{for all } a \text{ and } b \text{ in } G_1.$$

The notion of homomorphism is central to the study of groups and so we give some examples. Take $G_1 = \mathbb{Z}$ to be the integers and $G_2 = C_2$. Define the map ϕ by

$$\phi(n) = (-1)^n.$$

Recall that group 'multiplication' is ordinary addition in $\mathbb{Z}$ so that the condition that ϕ be a homomorphism reduces to the assertion that

$$\phi(a+b) = \phi(a)\phi(b),$$

i.e that

$$(-1)^{a+b} = (-1)^a(-1)^b$$

which is clearly true. More generally, we can define a homomorphism from $\mathbb{Z}$ to C_k by

$$\phi(a) = \exp 2\pi i a/k = \omega^a, \quad \text{where } \omega \text{ equals } \exp 2\pi i/k.$$

This generalizes the construction of Example 1 of the preceding section. Basically, what the homomorphism ϕ is telling us is that we can regard multiplication in C_k as 'addition modulo k' in the integers.

We now want to describe another homomorphism which has many important physical applications and which will recur frequently in the rest of this book. For this we need to introduce still another group, the Lorentz group. Let M denote the four-dimensional space $M = \mathbb{R}^4$, with the 'Lorentz metric'

$$\|\mathbf{x}\|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2, \quad \text{where } \mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

此为试读,需要完整PDF请访问: www.ertongbook.com