APPLIED STOCHASTIC METHODS SERIES

# Applied Diffusion Processes from Engineering to Finance


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# Introduction

The aim of this book is to facilitate interaction among engineering, finance and the insurance sectors as there are a lot of common models and solution methods for solving real-life problems in these three fields.

In the 19th Century, many problems in physics, for example heat diffusion, were theoretically solved using partial differential equations (PDEs). This led to new problems in mathematical analysis and later in function analysis; in particular, concerning the existence and unicity for the solutions of such PDE equations giving initial conditions of a Cauchy type that is the knowledge of the unknown function on a regular curve of the adequate Euclidean space.

Unfortunately, such PDEs have, in general, no explicit solution and so the problem of numerical treatment was posed. Although mathematicians could, indeed, formulate algorithms to give a good numerical approximation of the solution, it was nevertheless difficult to use such algorithms in practice, and it is only in the late 20th Century that this became possible with, of course, the building of more and more powerful computers together with elaborate software for numerical analysis.

In the 1970s, stochastic finance came into existence with the work of Black, Scholes and Merton, just after the fundamental results of Markowitz and Sharpe.

The main result is the celebrated Black and Scholes formula giving the value of a European call option with a closed formula. It can only be obtained by the authors with a laborious analytical transformation of their PDE arriving at the resolution of a well-known equation in physics called the diffusion equation.

Without this result, it is probable that the Black and Scholes formula would not exist.

So with partial differential equations as the vehicle, the interaction among engineering, physics and finance plays a fundamental role, and this book will show that this role is of major importance to get new results in finance and that, moreover, it could also be applied in the other spheres.

In Chapters 1–3, basic diffusion phenomena and models, probabilistic models of diffusion processes and the related PDEs including the heat equation are presented together with some fundamental results in stochastic calculus such as Itô's and Feynman–Kac's formulas.

Chapter 4 presents fundamental problems in finance concerning the stochastic evolution of stock prices and interest rates, while Chapter 5 shows how PDEs are necessary to price financial products, such as options and zero-coupon bonds, and how the interaction with PDE in physics works. It also shows that some important methods in finance, such as the use of the risk-neutral measure with Girsanov's theorem, are nothing other than the use of Green's function that is presented in Chapter 3.

Chapter 6 thoroughly analyzes stochastic finance with the consideration of more sophisticated derivative products than the plain vanilla European options (exotic options, American options, etc.). The pricing of such financial products is done with the resolution of PDEs with the methods of physics and engineering as presented in Chapter 3.

Chapter 7 presents some applications in stochastic insurance using hitting times for diffusion processes such as the Merton model for credit risk and asset liability models (ALM) to model the risk of banks and insurance companies.

Chapter 8 first describes the finite difference method and its application for solving numerically the PDEs of Black and Scholes as presented in the preceding chapters.

The next four chapters (Chapters 9–12) discuss some recent advanced topics such as non-linear problems, Lévy processes, and the copula approach and semi-Markov models in interaction with diffusion models.

This is, in particular, important for the evolution from Gaussian to non-Gaussian stochastic finance in future years as, indeed, recent crises imply considering the case of non-efficient and incomplete markets. In particular, this extension can be done with jumps models, generalizing the Merton model for option pricing. We can also use an economic-financial environmental process using semi-Markov theory. These last processes are also useful for pricing American options with a discrete-time model.

The last chapter (Chapter 13) presents some simulation results as it is a fact that for some real situations, there do not exist simple closed formulas and so simulation is the only possibility.

Finally, the Conclusion deals with actual and future interactions among engineering, finance and insurance as a fructuous source of developments for new models that are more adapted to approach the complexity of our three basic fields, thereby showing the great originality of this book.

This book is intended for a large audience of professional, research and academic disciplines, including engineers, mathematicians, physicists, actuaries and finance researchers with a good knowledge of probability theory.

# Table of Contents

# Chapter 1

# Diffusion Phenomena and Models

The aim of this chapter is to obtain the differential diffusion equation from the macroscopic point of view starting from a microscopic point of view. The approach is heuristic and a rigorous analysis is found in the current literature as also suggested in the following sections. The equation is obtained with reference to the mass diffusion phenomena and also by analogy to heat conduction. Then the analysis is carried out with reference to this last physical aspect. The parabolic and elliptic equations are presented and the initial and boundary conditions are also given.

In doing so, we can see in the following chapters why stochastic finance uses the results of diffusion theory.

## 1.1. General presentation of diffusion process

In general, a diffusion phenomenon is a process in which some physical properties are transported at molecular or atomic level from one part of the space to another part. The process is the result of random migration of small particles inside the physical space. It determines the motion of matter as well as energy. From a general point of view, the diffusion concept or phenomenon is also related to the random movement of small particles, and a very simple example is given by an observer on a skyscraper watching a crowded square: people move in all directions randomly but uniformly. Another example is a red wine drop in a glass filled with water. After some time the water becomes uniformly light pink in color. This suggests that the wine overruns the water, the molecules of wine are everywhere and the wine is said to have diffused into the water. This mass transport is due to the molecular agitation with the result that zones with a high concentration of wine determine a net molecular mass movement in all directions toward zones with lower wine concentration. In fact, an individual molecule of wine moves randomly and in a dilute solution each molecule of wine acts independently of the other molecules and undergoes collisions with the water molecules. The motion of a single molecule of wine can be depicted by the term of a "random walk" as shown in Figure 1.1. The picture of random molecular motions should adapt with the fact that a transfer of molecules from the region of higher concentration to the region of lower concentration is observed. If two thin zones are considered with equal volumes, one with a higher concentration and the other with a lower concentration, there is a dynamic exchange. A net transfer of molecules from the higher concentration to the lower concentration is obtained according to the second law of thermodynamics. Some other examples and descriptions are found in several books on this topic [BAK 08, CRA 75, CUS 09, GHE 88].

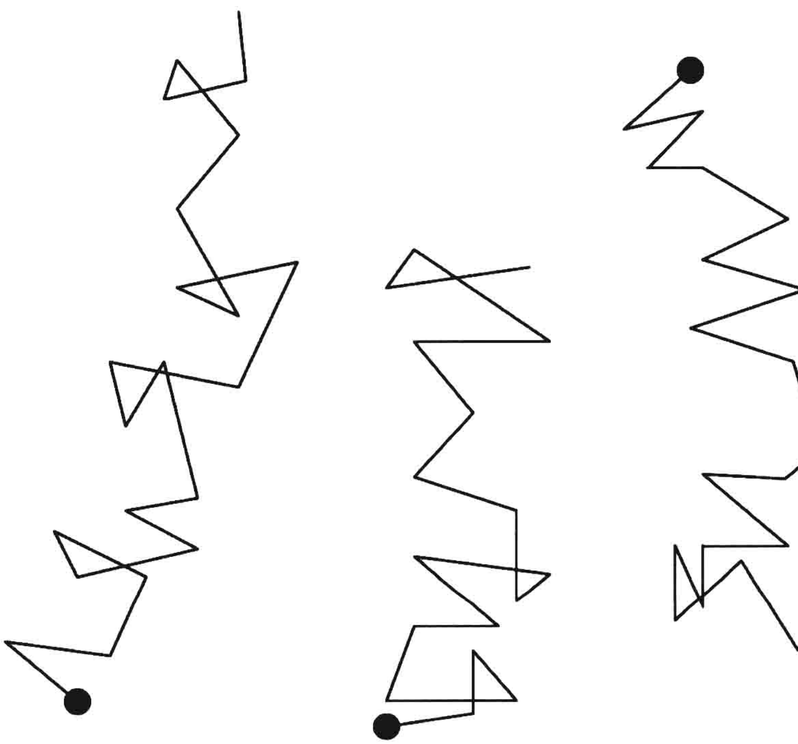

**Figure 1.1.** *Path of Brownian motion*

The molecular transfer determines a mass diffusion and, consequently, a diffusion of the other physical properties, such as the energy or more precisely an energy flux in conduction mode, is present. It needs to describe mathematically the molecular random transfer and to obtain a macroscopic description by means of a