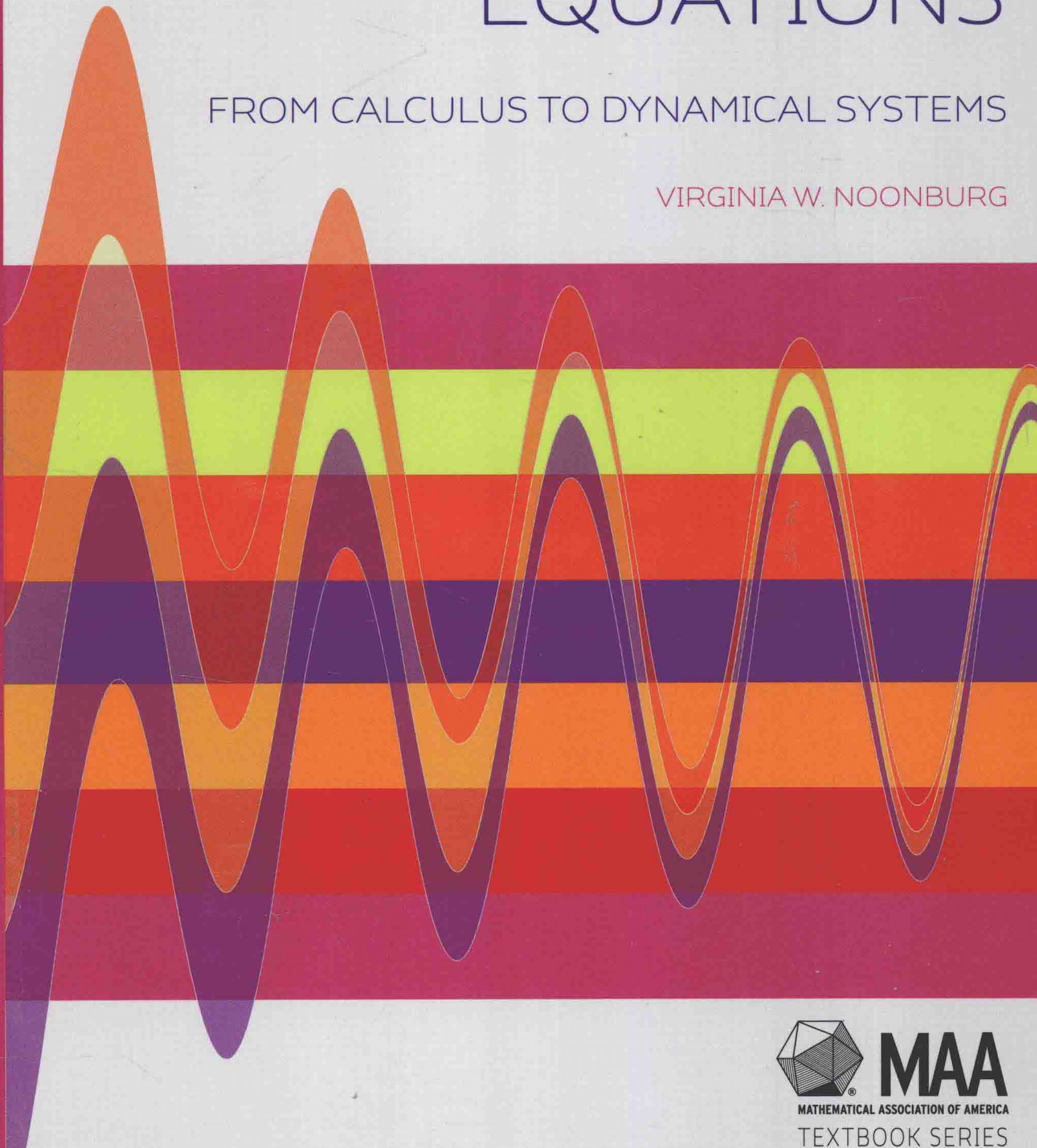
ORDINARY DIFFERENTIAL EQUATIONS

FROM CALCULUS TO DYNAMICAL SYSTEMS

VIRGINIA W. NOONBURG



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Ordinary Differential Equations
from Calculus to Dynamical Systems

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Preface

This is, first and foremost, a text for the introductory course in ordinary differential equations, usually taken by sophomore engineering and science majors after a two or three term calculus sequence. The driving idea behind this particular book is that if all science majors can be convinced to take the differential equations course along with the engineers, they will be in a much better position if they go on to graduate school or even if they just want to read the modern literature in their own field. An understanding of dynamical systems is gradually becoming a necessity even for those in the biological sciences.

Much of my own research is in mathematical biology, and I know that there are new and interesting problems out there to be solved. In looking at several texts used for teaching courses in mathematical biology, I found that they tend to assume very little in the way of a mathematical background. This would not be the case if these students could be encouraged to take a course in differential equations early in their career. One possible way to accomplish this is to show students that problems involving differential equations are often a lot more interesting than the problems seen in calculus. This is especially true in the area of nonlinear equations, and is one reason why this book contains much more than the usual amount of material on the geometry of nonlinear systems.

It is also my hope that getting engineers and applied scientists working together on interesting problems at this level will lead to more joint projects later on, both in graduate school and in industry. As an example of this type of cooperation, the final project in Chapter 5 is based on a recent research paper written jointly by a mathematician and a biologist.

Since not all students are fortunate enough to take a differential equations course in their undergraduate career, another aim has been to make this book as readable as possible so it can be used for self-study. Readability and worked-out answers to problems also makes this text a likely candidate for a professor who wants to teach a "flipped" course in differential equations.

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I also want to take this opportunity to acknowledge the constant help and encouragement from my husband Bill, a Cornell engineer, whose insightful feedback kept me from straying too far from the engineering origins of my subject.

Sample Course Outline

At the University of Hartford, classes in differential equations (taught in the Mathematics Department) generally contain about 70–80% engineering students. They are mostly sophomores who have just completed a two-term sequence in calculus (including a brief introduction to separable equations), have had no formal exposure to linear algebra, and are expected to gain a reasonable familiarity with Laplace transforms in the differential equations course. The other 20–30% of the class is a mix of math and science students, and there may even be a few music students who are interested in acoustics.

Most of the classes meet for 75 minutes, twice a week, during a term of approximately 14 weeks. A representative schedule for such a course is shown below.

Week	Material Covered
1	Chap. 1, 2.1
	↓
2	2.2, 2.3
	↓
3	2.4, 2.6
	↓
4	2.7, project
	↓
5	3.1, 3.2
	↓
6	3.3, 3.4
	↓
7	3.5, 3.7
	↓
8	4.1, 4.2
	↓
9	4.3, 4.4
	↓
10	5.1, 5.2
	↓
11	5.3, project
	↓
12	6.1, 6.2
	↓
13	6.3, 6.4
	↓
14	6.5, project

If Laplace transforms is not a required topic in the course, much more time can be spent on Chapter 5. It also gives the students more time to absorb the material on linear algebra in Chapter 4.

There are three sections in the book that are not prerequisites for any other sections: 2.5 (More Analytic Methods), 3.6 (Linear Second-order Equations with Non-constant Coefficients), and 4.5 (The Matrix Exponential).

Contents

CHAPTER 1

Introduction to Differential Equations

Differential equations arise from real-world problems and problems in applied mathematics. One of the first things you are taught in calculus is that the derivative of a function is the instantaneous rate of change of the function with respect to its independent variable. When mathematics is applied to real-world problems, it is often the case that finding a relation between a function and its rate of change is easier than finding a formula for the function itself; it is this relation between an unknown function and its derivatives that produces a differential equation.

To give a very simple example, a biologist studying the growth of a population, with size at time t given by the function $P(t)$, might make the very simple, but logical, assumption that a population grows at a rate directly proportional to its size. In mathematical notation, the equation for $P(t)$ could then be written as

$$\frac{dP}{dt} = rP(t),$$

where the constant of proportionality, r, would probably be determined experimentally by biologists working in the field. Equations used for modeling population growth can be much more complicated than this, sometimes involving scores of interacting populations with different properties; however, almost any population model is based on equations similar to this.

In an analogous manner, a physicist might argue that all the forces acting on a particular moving body at time t depend only on its position $x(t)$ and its velocity $x'(t)$. He could then use Newton's second law to express mass times acceleration as $mx''(t)$ and write an equation for $x(t)$ in the form

$$mx''(t) = F(x(t), x'(t)),$$

where F is some function of two variables. One of the best-known equations of this type is the spring-mass equation

$$mx'' + bx' + kx = f(t), \tag{1.1}$$

in which $x(t)$ is the position at time t of an object of mass m suspended on a spring, and b and k are the damping coefficient and spring constant, respectively. The function f represents an external force acting on the system. Notice that in (1.1), where x is a function of a single variable, we have used the convention of omitting the independent variable t, and have written x, x', and x'' for $x(t)$ and its derivatives.

In both of the examples, the problem has been written in the form of a differential equation, and the solution of the problem lies in finding a function $P(t)$, or $x(t)$, which makes the equation true.

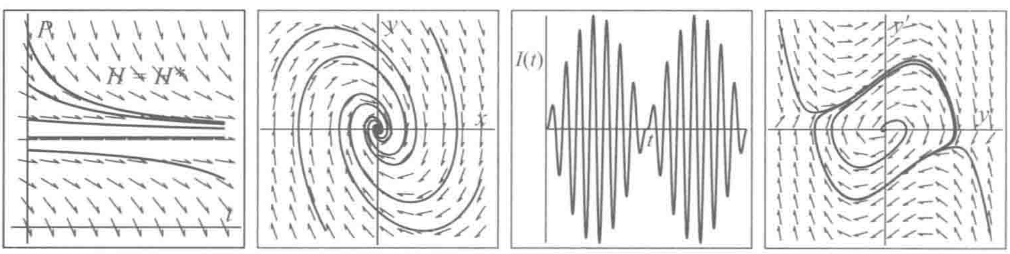
1.1 Basic Terminology

Before beginning to tackle the problem of formulating and solving differential equations, it is necessary to understand some basic terminology. Our first and most fundamental definition is that of a differential equation itself.

Definition 1.1. A **differential equation** is any equation involving an unknown function and one or more of its derivatives.

The following are examples of differential equations:

1. $P'(t) = rP(t)(1 - P(t)/N) - H$ harvested population growth
2. $\frac{d^2x}{d\tau^2} + 0.9\frac{dx}{d\tau} + 2x = 0$ spring-mass equation
3. $I''(t) + 4I(t) = \sin(\omega t)$ RLC circuit showing "beats"
4. $y''(t) + \mu(y^2(t) - 1)y'(t) + y(t) = 0$ van der Pol equation
5. $\frac{\partial^2}{\partial x^2}u(x, y) + \frac{\partial^2}{\partial y^2}u(x, y) = 0$ Laplace's equation



For the first four equations the graphs above illustrate different ways of picturing the solution curves.

1.1.1 Ordinary vs. Partial Differential Equations

Differential equations fall into two very broad categories, called ordinary differential equations and partial differential equations. If the unknown function in the equation is a function of only one variable, the equation is called an **ordinary differential equation**. In the list of examples, equations 1-4 are ordinary differential equations, with the unknown functions being $P(t)$, $x(\tau)$, $I(t)$, and $y(t)$ respectively. If the unknown function in the equation depends on more than one independent variable, the equation is called a **partial differential equation**, and in this case, the derivatives appearing in the equation will be partial derivatives. Equation 5 is an example of an important partial differential equation, called Laplace's equation, which arises in several areas of applied mathematics. In equation 5, u is a function of the two independent variables x and y. In this book, we will not consider