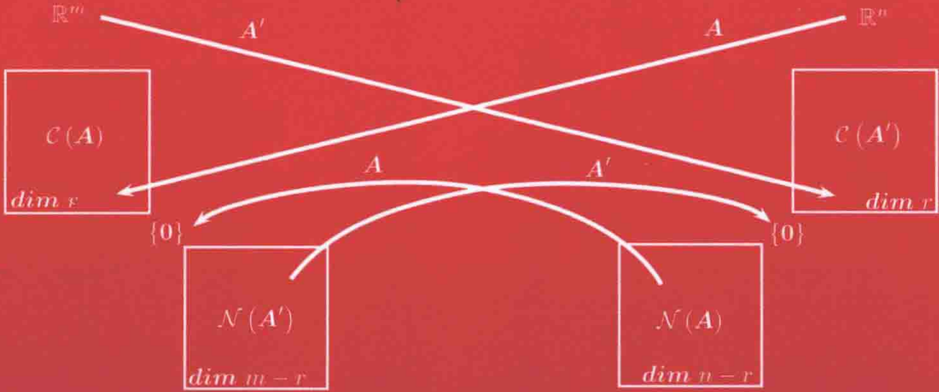
Texts in Statistical Science

Linear Algebra and Matrix Analysis for Statistics



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Texts in Statistical Science

Linear Algebra and Matrix Analysis for Statistics

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To my parents, Shyamali and Sunit, my wife, Sharbani, and my son, Shubho.

—Sudipto Banerjee

To my wife, Ishani.

—Anindya Roy

Preface

Linear algebra constitutes one of the core mathematical components in any modern curriculum involving statistics. Usually students studying statistics are expected to have seen at least one semester of linear algebra (or applied linear algebra) at the undergraduate level. In particular, students pursuing graduate studies in statistics or biostatistics are expected to have a sound conceptual grasp of vector spaces and subspaces associated with matrices, orthogonality, projections, quadratic forms and so on.

As the relevance and attraction of statistics as a discipline for graduate studies continues to increase for students with more diverse academic preparations, the need to accommodate their mathematical needs also keeps growing. In particular, many students find their undergraduate preparation in linear algebra rather different from what is required in graduate school. There are several excellent texts on the subject that provide as comprehensive a coverage of the subject as possible at the undergraduate level. However, some of these texts cater to a broader audience (e.g., scientists and engineers) and several formal concepts that are important in theoretical statistics are not emphasized.

There are several excellent texts on linear algebra. For example, there are classics by Halmos (1974), Hoffman and Kunze (1984) and Axler (1997) that make heavy use of vector spaces and linear transformations to provide a coordinate-free approach. A remarkable feature of the latter is that it develops the subject without using determinants at all. Then, there are the books by Strang (2005, 2009) and Meyer (2001) that make heavy use of echelon forms and canonical forms to reveal the properties of subspaces associated with a matrix. This approach is tangible, but may not turn out to be the most convenient to derive and prove results often encountered in statistical modeling. Among texts geared toward statistics, Rao and Bhimsankaram (2000), Searle (1982) and Graybill (2001) have stood the test of time. The book by Harville (1997) stands out in its breadth of coverage and is already considered a modern classic. Several other excellent texts exist for statisticians including Healy (2000), Abadir and Magnus (2005), Schott (2005) and Gentle (2010). The concise text by Bapat (2012) is a delightful blend of linear algebra and statistical linear models.

While the above texts offer excellent coverage, some expect substantial mathematical maturity from the reader. Our attempt here has been to offer a more gradual exposition to linear algebra without really dumbing down the subject. The book tries to be as self-contained as possible and does not assume any prior knowledge of linear algebra. However, those who have seen some elementary linear algebra will be able to move more quickly through the early chapters. We have attempted to present both the vector space approach as well as the canonical forms in matrix theory.

Although we adopt the vector space approach for much of the later development, the book does not begin with vector spaces. Instead, it addresses the rudimentary mechanics of linear systems using Gaussian elimination and the resultant decompositions (Chapters 1–3). Chapter 4 introduces Euclidean vector spaces using less abstract concepts and makes connections to systems of linear equations wherever possible. Chapter 5 is on the rank of a matrix. Why devote an entire chapter to rank? We believe that the concept of rank is *that* important for a thorough understanding. In several cases we show how the same result may be derived using multiple techniques, which, we hope, will offer insight into the subject and ensure a better conceptual grasp of the material. Chapter 6 introduces complementary subspaces and oblique projectors. Chapter 7 introduces orthogonality and orthogonal projections and leads us to the Fundamental Theorem of Linear Algebra, which connects the four fundamental subspaces associated with a matrix. Chapter 8 builds upon the previous chapter and focuses on orthogonal projectors, which is fundamental to linear statistical models, and also introduces several computational techniques for orthogonal reduction. Chapter 9 revisits linear equations from a more mature perspective and shows how the theoretical concepts developed thus far can be handy in analyzing solutions for linear systems. The reader, at this point, will have realized that there is much more to linear equations than Gaussian elimination and echelon forms. Chapter 10 discusses determinants. Unlike some classical texts, we introduce determinants a bit late in the game and present it as a useful tool for characterizing and obtaining certain useful results. Chapter 11 introduces eigenvalues and eigenvectors and is the first time complex numbers make an appearance. Results on general real matrices are followed by those of real symmetric matrices. The popular algorithms for eigenvalues and eigenvectors are outlined both for symmetric and unsymmetric matrices. Chapter 12 derives the Singular value decomposition and the Jordan Canonical Form and presents an accessible proof of the latter. Chapter 13 is devoted to quadratic forms, another topic of fundamental importance to statistical theory and methods. Chapter 14 presents Kronecker and Hadamard products and other related materials that have become conspicuous in multivariate statistics and econometrics. Chapters 15 and 16 provide a taste of some more advanced topics but, hopefully, in a more accessible manner than more advanced texts. The former presents some aspects of linear iterative systems and convergence of matrices, while the latter introduces more general vector spaces, linear transformations and Hilbert spaces.

We remark that this is not a book on matrix computations, although we describe several numerical procedures in some detail. We have refrained from undertaking a thorough exploration of the most numerically stable algorithms as they would require a lot more theory and be too much of a digression. However, readers who grasp the material provided here should find it easier to study more specialized texts on matrix computations (e.g., Golub and Van Loan, 2013; Trefthen and Bau III, 1997). Also, while we have included many exercises that can be solved using languages such as

MATLAB and R, we decided not to marry the text to a specific language or platform. We have also not included statistical theory and applications here. This decision was taken neither in haste nor without deliberation. There are plenty of excellent texts on the theory of linear models, regression and modeling that make abundant use of linear algebra. Our hope is that readers of this text will find it easier to grasp the material in such texts. In fact, we believe that this book can be used as a companion text in the more theoretical courses on linear regression or, perhaps, stand alone as a one-semester course devoted to linear algebra for statistics and econometrics.

Finally, we have plenty of people to thank for this book. We have been greatly influenced by our teachers at the Indian Statistical Institute, Kolkata. The book by Rao and Bhimsankaram (2000), written by two of our former teachers whose lectures and notes are still vivid in our minds, certainly shaped our preparation. Sudipto Banerjee would also like to acknowledge Professor Alan Gelfand of Duke University with whom he has had several discussions regarding the role of linear algebra in Bayesian hierarchical models and spatial statistics. The first author also thanks Dr. Govindan Rangarajan of the Indian Institute of Science, Bangalore, India, Dr. Anjana Narayan of California Polytechnic State University, Pomona, and Dr. Mohan Delampady of Indian Statistical Institute, Bangalore, for allowing the author to work on this manuscript as a visitor in their respective institutes. We thank the Division of Biostatistics at the University of Minnesota, Twin Cities, and the Department of Statistics at the University of Maryland, Baltimore, for providing us with an ambience most conducive to this project. Special mention must be made of Dr. Rajarshi Guhaniyogi, Dr. Joao Monteiro and Dr. Qian Ren, former graduate students at the University of Minnesota, who have painstakingly helped with proof-reading and typesetting parts of the text. This book would also not have happened without the incredible patience and cooperation of Rob Calver, Rachel Holt, Sarah Gelson, Kate Gallo, Charlotte Byrnes and Shashi Kumar at CRC Press/Chapman and Hall. Finally, we thank our families, whose ongoing love and support made all of this possible.

SUDIPTO BANERJEE — Minneapolis, Minnesota
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