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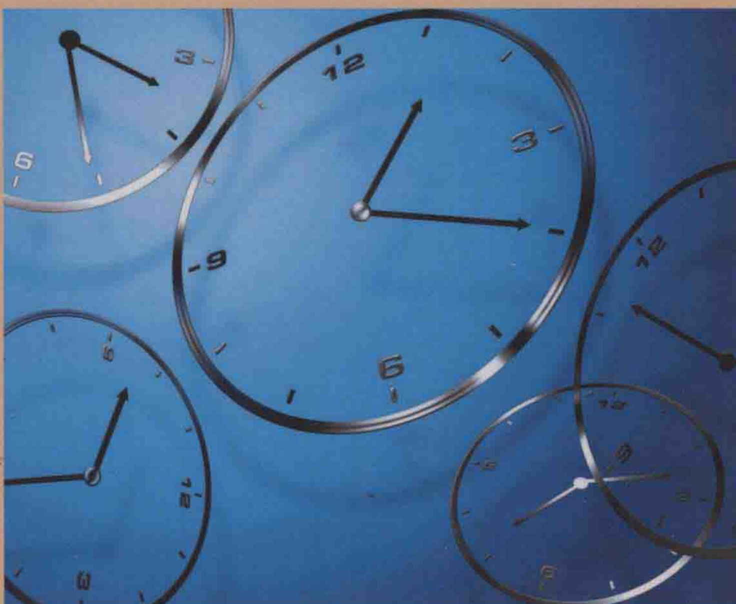
CHANGE OF TIME AND CHANGE OF MEASURE

Second Edition



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World Scientific

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Foreword to the Second Edition

The only change to the First Edition of the present book is that an additional Chap. 13 has been added that outlines developments in the topics of the book that have taken place since the publication of the First Edition. These developments have mainly arisen out of studies of the statistical theory of turbulence, but they encompass also results and applications to financial econometrics. The new material falls within the recently established field termed Ambit Stochastics. Some of the topics not discussed in the original Edition are random measures and Lévy bases, metatimes (a multivariate form of timechange), change of Lévy measures, and the new classes of processes termed Brownian semistationary (or BSS) processes and Lévy semistationary (or LSSD) processes. As in the former part of the book, the concepts of volatility/intermittency play a central role.

Foreword

The conception of the book, based on LECTURE COURSES delivered by the authors in the last years (Aarhus, Moscow, Barcelona, Halmstad, *etc.*), is defined in many respects by the desire to state the main ideas and results of the stochastic theory of "change of time and change of measure". These ideas and results have manifold applications, particularly in Mathematical Finance, Financial Economics, Financial Engineering and Actuarial Business, when constructing probabilistic and statistical models adequate to statistical data, when investigating the problems of arbitrage, hedging, rational (fair) pricing of financial and actuarial instruments, when making decisions minimizing the financial and actuarial risks, *etc.* The lecture-based character of the book defined as well the style of presentation—we have not aimed to give all and complete proofs, many of which are rather long. Our purpose was different, namely to specify the main, essential topics and results of "change of time and change of measure", so that the readers could make use of them in their theoretical and applied activity.

Acknowledgments. We express our gratitude to our colleagues, especially Ernst Eberlein and Neil Shephard, for stimulating discussions. We are grateful to the Thiele Centre (Department of Mathematical Sciences, Aarhus University) and the Steklov Mathematical Institute (Moscow) for providing excellent opportunities to work on the monograph. The support of INTAS, RFBR, Manchester University (School of Mathematics), and Moscow State University (Department of Mechanics and Mathematics) is gratefully acknowledged. We thank T. B. Tolozova for her help in preparation of the text for publication.

O. E. B.-N.,
A. N. Sh.

Introduction

1. One of the topical problems of Probability Theory and the Theory of Stochastic Processes is the following:

> How, for the given stochastic processes (maybe with "complicated" structure), to obtain a relatively simple representation via some "simple" processes of the type of "white noise" in discrete-time case or Brownian motion or Lévy processes in the case of continuous time?

For example, from the theory of stationary sequences we know that every "regular" sequence $X = (X_n)$ admits the "Wold decomposition"

$$X_n = \sum_{k=0}^{\infty} a_k \varepsilon_{n-k}, \qquad n \in \mathbb{Z} = \{\dots, -1, 0, 1, \dots\},$$

where $\varepsilon = (\varepsilon_n)$ is a sequence of pairwise orthogonal random variables ("white noise") with $\mathsf{E}\varepsilon_n = 0$, $\mathsf{E}|\varepsilon_n|^2 = 1$.

Another example. If we agree that the Brownian motion $B = (B_t)_{t \geq 0}$ is a process of a "simple" structure, then a solution $X = (X_t)_{t \geq 0}$ to the Itô stochastic differential equation

$$dX_t = a(t, X_t)\, dt + \sigma(t, X_t)\, dB_t$$

can be considered as a version of (a candidate for) the Kolmogorov diffusion process with the local characteristics $a(t, x)$ and $\sigma(t, x)$.

In the present book our main interest will be related to the following two methods for getting "simple" representations:

CHANGE OF TIME and *CHANGE OF MEASURE.*

We also shall consider another method of representation of the processes based on stochastic integrals with respect to some "simple" processes. This method is convenient as an intermediate step for getting the change of time representation for the "complicated" processes.

The *change of time* is based on the idea of representation of a given process $X = (X_t)_{t\geq 0}$ *via* a "simple" process $\widehat{X} = (\widehat{X}_\theta)_{\theta\geq 0}$ and a "change of time" $T = (T(t))_{t\geq 0}$:

$$X = \widehat{X} \circ T$$

or, in detail, $X_t = \widehat{X}_{T(t)}$. In other words, the process X is a time-deformation of the process $\widehat{X}$. This can be considered as a way to change the velocity in moving along the trajectories of $\widehat{X}$.

The technique of *change of measure* does not operate with the transformation of the trajectories. Instead it is based on the construction of a new probability measure $\widetilde{\mathsf{P}}$ equivalent to the given measure P and a process $\widetilde{X} = (\widetilde{X}_t)_{t\geq 0}$ with a "simple" structure such that

$$\mathrm{Law}(X \mid \widetilde{\mathsf{P}}) = \mathrm{Law}(\widetilde{X} \mid \mathsf{P}).$$

From the point of view of applications the general problem of *change of measure* is of central interest in mathematical finance, where so-called "martingale measures" $\widetilde{\mathsf{P}}$ play a key role for criteria of "No-Arbitrage" and for the "Pricing and Hedging" machinery.

The concept of *change of time* is also of substantial interest for understanding the nature of financial time series; witness the common phrase that "Prices on the financial markets are Brownian motions in the operational (or business) time".

2. Let us give a more detailed description of the content of the chapters of the book.

Chapter 1 contains some material about Brownian motion and Lévy processes as main "simple" driving processes used in constructing the change of time. Because these important processes and the processes constructed from them usually belong to the class of semimartingales, we have included also some text (Chap. 3) about semimartingales which become more and more popular in many fields and in mathematical finance in particular.

The general scheme of change of time ("old time" → "new time" → "old time" discussed in Chap. 1 is the following.

Assume a stochastic process $X = (X_t)_{t\geq 0}$ to be given (in "old" time t) on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathsf{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the "flow" of information ($\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $s \leq t$). We construct an increasing family $\widehat{T} = (\widehat{T}(\theta))_{\theta\geq 0}$ of stopping times $\widehat{T}(\theta)$ (with respect to $\mathbb{F}$)

and we introduce a "new" process $\widehat{X} = (\widehat{X}_\theta)_{\theta \geq 0}$ (in a "new" time θ) by the formula

$$\widehat{X}_\theta = X_{\widehat{T}(\theta)}$$

which we shall usually write in a short form

$$\widehat{X} = X \circ \widehat{T}.$$

Suppose that the process $\widehat{X}$ has a simple structure, then it is reasonable to try to find a new increasing family of stopping times $T = (T(\theta))_{\theta \geq 0}$ (with respect to $\widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_\theta)_{\theta \geq 0}$, where $\widehat{\mathcal{F}}_\theta = \mathcal{F}_{\widehat{T}(\theta)}$), such that the following representation in the *strong* sense holds:

$$X = \widehat{X} \circ T, \qquad i.e., \qquad X_t = \widehat{X}_{T(t)}, \quad t \geq 0.$$

The given description distinguishes between "old" ("physical", "calendar") time t and a "new" ("operational", "business") time θ, with $\theta = T(t)$ and $t = \widehat{T}(\theta)$ as the formulae which define the transitions: $t \to \theta \to t$.

All previous considerations had the following aim: given an "old" (initial) process X (in time t), to construct a simple "new" process $\widehat{X}$ (in time θ) and to construct two changes of time $\widehat{T}(\theta)$ and $T(t)$ such that X and $\widehat{X}$ can be obtained by the transformations $\widehat{X} = X \circ \widehat{T}$ and $X = \widehat{X} \circ T$.

So far we have assumed that the property $\widehat{X} = X \circ \widehat{T}$ (and $X = \widehat{X} \circ T$) holds identically (for all $\omega \in \Omega$ and all $t \geq 0$). However, sometimes such representations are hard to find but one can find representations of the type $X \stackrel{\text{a.s.}}{=} \widehat{X} \circ T$ (so-called *semi-strong representations*) or $X \stackrel{\text{law}}{=} \widehat{X} \circ T$ (so-called *weak representations*).

Chapter 2 is about

STOCHASTIC VOLATILITY REPRESENTATION or
STOCHASTIC INTEGRAL REPRESENTATION

$$\boxed{X = H \cdot \widetilde{X}}$$

given the process X, where $H \cdot \widetilde{X}$ is the *stochastic integral* with respect to some "simple" process $\widetilde{X}$ (usually assumed to be a semimartingale); the integrand H is often called a *stochastic volatility.*

Having the

CHANGE OF TIME REPRESENTATION

$$\boxed{X = \widehat{X} \circ T}$$

of the process X *via* some "simple" process $\widehat{X}$ and a *change of time* T we get very transparent connection between the stochastic volatility models and the change of time models:

$$\boxed{H \cdot \widetilde{X} = \widehat{X} \circ T}\,.$$

We emphasize that this duality of the "volatility" and the "change of time" plays a very important role in the construction of convenient models. Especially it is important for the financial modeling.

For many popular models in finance the processes $\widetilde{X}$ are Brownian motions or Lévy processes. So, stochastic volatility models with semimartingales $\widetilde{X}$ cover the most commonly used models.

It is useful to note that to define the time-changed process $\widehat{X} \circ T$ we need not assume that $\widehat{X}$ is a semimartingale.

In Chap. 2 we also consider more general stochastic integral representations (using measures of jumps). As in Chap. 1, both "strong" and "weak" representations are discussed.

Chapter 3 contains important material about *semimartingales*, *i.e.*, stochastic processes $X = (X_t)_{t \geq 0}$ representable as sums $X = X_0 + A + M$, where $A = (A_t)_{t \geq 0}$ is a process of bounded variation and $M = (M_t)_{t \geq 0}$ is a local martingale. This class is rather wide, the stochastic calculus for these processes is well developed, and they proved to be useful for the study of problems in mathematical finance, actuarial mathematics, and many other fields.

Without any doubt, the class of semimartingale models, including those of Brownian and Lévy type, will play the increasingly important roles in applications of stochastic calculus, not least in finance.

In *Chap. 4* some fundamental notions, namely, stochastic exponential, stochastic logarithm, and cumulant processes, are introduced. These will be of high importance in the rest of the monograph.

Chapter 5 provides a short survey of processes with independent increments (PII), in particular of *Lévy processes*. In some sense the class of semimartingales is a very natural extension of the Lévy processes. Indeed, for PII the triplet (B, C, ν) of characteristics, involved in the Kolmogorov–Lévy–Khinchin formula for the processes with independent increments, consists of *deterministic* components. In the case of semimartingales there exists also a similar triplet (B, C, ν) whose components have the *predictability* property which can be interpreted as a *stochastic determinancy*.

Change of measure plays a crucial role in both probability theory and its applications, providing a powerful tool for study of the distributional properties of stochastic processes. *Chapter 6* "Change of Measure. General Facts" serves as a quick introduction to this subject.

In *Chap. 7* we focus on problems of change of measure especially for Lévy processes, which constitute now a basis for construction of different models (in finance, actuarial science, *etc.*).

Chapter 8 is devoted to the other (along with change of measure) key topic of the book, namely, change of time in semimartingale, Brownian, and Lévy models.

Chapter 9 plays an important conceptual role in our monograph. Firstly, this chapter reviews the "martingale-predictable" approach (based on Doob's decomposition) to study of sequences $H = (H_n)_{n \geq 0}$ which describe the evolution of financial indexes $S_n = S_0 e^{H_n}$, $n \geq 0$, and "return" sequences $h = (h_n)_{n \geq 0}$, where $h_n = \log(S_n/S_{n-1})$ ($\equiv \Delta H_n = H_n - H_{n-1}$). This "martingale-predictable" scheme naturally comprises both linear (AR, MA, ARMA, *etc.*) and nonlinear (ARCH, GARCH, *etc.*) models.

Secondly, in this chapter we introduce the class of $\mathbb{G}$IG ($\mathbb{G}$eneralized Inverse Gaussian) distributions for σ^2, the square of stochastic volatility, and the class of $\mathbb{G}$H ($\mathbb{G}$eneralized Hyperbolic) distributions for the "returns" $h = \mu + \beta\sigma^2 + \sigma\varepsilon$, where σ and ε are independent, σ^2 has $\mathbb{G}$IG-distribution, and ε has the standard Gaussian distribution $\mathcal{N}(0, 1)$. The most recent econometric investigations show convincingly that $\mathbb{G}$IG and $\mathbb{G}$H distributions fit well the empirical distributions of various financial indexes.

Chapter 10 demonstrates, first of all, how ideas of change of measure allow one to transform the economic notion of arbitrage into the martingale property of (normalized) prices with respect to special measures called "martingale measures". We consider both discrete and continuous time cases.

Chapter 11 provides a short overview of basic results in the option pricing theory. We cite some classical formulae (Bachelier, Black–Scholes, Cox–Ross–Rubinstein) and discuss different properties such as call–put parity and call–put duality in the semimartingale and especially Lévy's models.

Chapter 12 is closely related to the material of Chap. 9. Since $\mathbb{G}$IG and $\mathbb{G}$H distributions, introduced in Chap. 9, are infinitely divisible, one can construct Lévy's processes $T = T(t)$, $t \geq 0$, and $H = (H_t)_{t \geq 0}$, such that $\mathrm{Law}(T(1)) = \mathbb{G}\mathrm{IG}$ and $\mathrm{Law}(H_1) = \mathbb{G}\mathrm{H}$. The process $H = (H_t)_{t \geq 0}$, can be chosen as $H_t = \mu + \beta T(t) + B_{T(t)}$, where $B = (B_t)_{t \geq 0}$ is a Brownian motion which does not depend on $T = (T(t))_{t \geq 0}$. Introduction of these processes (Sec. 12.3) is preceded by a review of a number of classical and modern financial models accentuated on stylized features of observed prices. Different types of models having desirable features are listed in Sec. 12.4.

Concluding the Introduction, we notice that the thorough reading of certain chapters demands sometimes a look into subsequent chapters. For example, already in Chap. 1 we mention stochastic integrals with respect to the Brownian motion (Wiener process) and semimartingales, although the careful construction of these integrals is given only in Chap. 2. In the same way, examples of Chap. 1 operate with hyperbolic and generalized hyperbolic distributions, whose detailed discussion is postponed to Chap. 9. We hope that this will not create any difficulty for the reader.

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