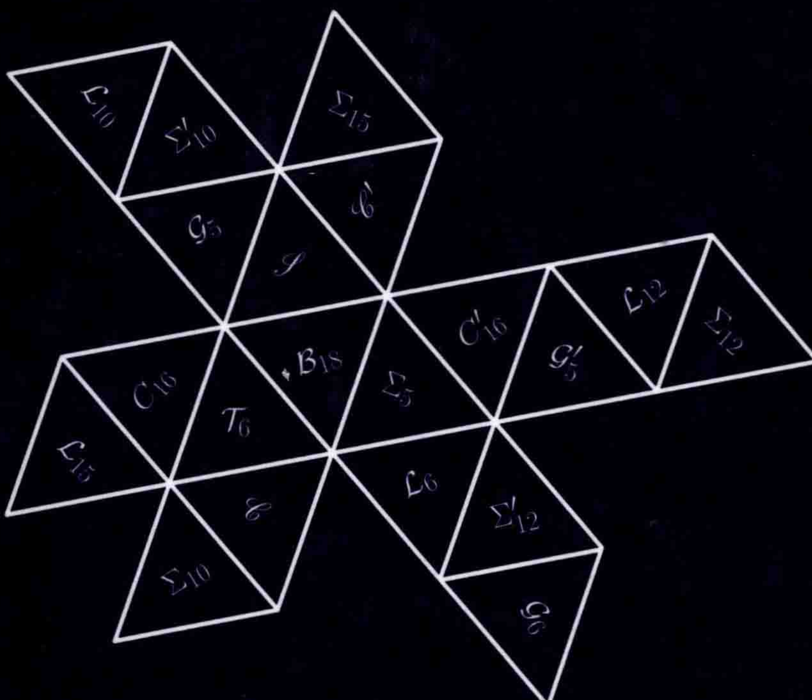
MONOGRAPHS AND RESEARCH NOTES IN MATHEMATICS

Cremona Groups and the Icosahedron



Ivan Cheltsov
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Preface

A complex projective space $\mathbb{P}^n$ is one of the most fundamental objects of algebraic geometry. It provides a motivation for the study of an exceptionally complicated object, the group $\mathrm{Cr}_n(\mathbb{C})$ of its birational automorphisms, called a *Cremona group* of rank n. This book deals with the Cremona group $\mathrm{Cr}_3(\mathbb{C})$ of rank 3, describing the beautiful appearances of the icosahedral group $\mathfrak{A}_5$ in it.

Most questions about the group $\mathrm{Cr}_1(\mathbb{C})$ are easy to answer, because it coincides with the group of biregular selfmaps of the projective line, which is isomorphic to $\mathrm{PGL}_2(\mathbb{C})$. The group $\mathrm{Cr}_2(\mathbb{C})$ is more complicated, since it does contain non-biregular transformations. The first example of such transformation, a circle inversion, was used by Apollonius of Perga to find a circle that is tangent to three given circles. This group has been intensively studied over the last two centuries, and many facts about it were established. These range from classical results about generators of $\mathrm{Cr}_2(\mathbb{C})$ due to Noether and Castelnuovo, and about relations between these generators due to Gizatullin, up to the action of $\mathrm{Cr}_2(\mathbb{C})$ on an infinite-dimensional hyperbolic space due to Cantat and Lamy, and complete classification of finite subgroups due to Dolgachev and Iskovskikh.

The structure of the group $\mathrm{Cr}_3(\mathbb{C})$ becomes way more complicated. It is known that it does not admit any "reasonable" set of generators. This group still resists any attempts to study its "global" structure, but one can access it on the level of finite subgroups, which became possible thanks to recent achievements in three-dimensional birational geometry. This accessibility is based on a general observation that a birational action of a finite group G on the projective space can be regularized, that is, replaced by a *regular* action of this group on some more complicated rational variety. This transfers the discussion into a rich world of varieties with large groups of symmetries.

At the moment it seems hardly possible to obtain a complete classification of finite subgroups in the Cremona group of rank 3. Nevertheless, Prokhorov managed to find all finite *simple* subgroups of $\mathrm{Cr}_3(\mathbb{C})$. He proved that the six groups $\mathfrak{A}_5$, $\mathrm{PSL}_2(\mathbf{F}_7)$, $\mathfrak{A}_6$, $\mathrm{SL}_2(\mathbf{F}_8)$, $\mathfrak{A}_7$, and $\mathrm{PSp}_4(\mathbf{F}_3)$ are the only non-abelian finite simple subgroups of $\mathrm{Cr}_3(\mathbb{C})$. The former three of these six groups actually admit embeddings to $\mathrm{Cr}_2(\mathbb{C})$, and the group $\mathfrak{A}_5$ is also realized as a subgroup of $\mathrm{PGL}_2(\mathbb{C})$, while the latter three groups are new three-dimensional artefacts.

Given a subgroup G of $\mathrm{Cr}_3(\mathbb{C})$ (or of any other group) it is natural to ask how many non-conjugate subgroups isomorphic to G are contained in the group $\mathrm{Cr}_3(\mathbb{C})$. It appears that methods of birational geometry fit very well to answer such questions. They allow classifying all embeddings of $\mathrm{SL}_2(\mathbf{F}_8)$, $\mathfrak{A}_7$, and $\mathrm{PSp}_4(\mathbf{F}_3)$ into $\mathrm{Cr}_3(\mathbb{C})$ up to conjugation (and actually there are very few of them). As a next step one can construct many non-conjugate embeddings of $\mathrm{PSL}_2(\mathbf{F}_7)$ and $\mathfrak{A}_6$ into $\mathrm{Cr}_3(\mathbb{C})$, although a complete answer is not known. The last remaining case that has not been studied yet is the smallest non-abelian simple group, the icosahedral group $\mathfrak{A}_5$, which is remarkable on its own. This book grew out of an attempt to fill this gap.

Being a group-theoretic counterpart of the icosahedron, the most symmetric of Platonic solids, the group $\mathfrak{A}_5$ may boast one of the longest histories of appearances in many areas of mathematics. A recognition of its significance is a famous book by Klein, centered mostly around this group. In connection with the discussed problem it behaves in an interesting way as well. On the one hand, there are many rational threefolds admitting icosahedral symmetry, including the projective space $\mathbb{P}^3$ itself, the three-dimensional quadric, and also classically studied Segre cubic, Igusa and Burkhardt quartics, and the double cover of $\mathbb{P}^3$ branched along Barth sextic surface. On the other hand, it is currently unknown whether the corresponding icosahedral subgroups of $\mathrm{Cr}_3(\mathbb{C})$ are conjugate or not. Moreover, even the most powerful method to study questions of this kind, the theory of birational rigidity, is usually not applicable here. One of the goals of our book is to expand the frontiers of its applicability, and in particular to present an example of an $\mathfrak{A}_5$-birationally rigid rational threefold.

At this point the second main character of the book enters the scene. Among the rational threefolds with an action of the icosahedral group there is one remarkable smooth variety, a quintic del Pezzo threefold V_5. Apart from having a rich group of symmetries, it has been studied from many points of view such as explicit birational transformations, Kähler–Einstein metrics, exceptional collections in derived categories and instanton bundles. However, its $\mathfrak{A}_5$-equivariant geometry has never been explored. We focus on this problem. We manage to describe explicitly a huge number of interesting $\mathfrak{A}_5$-invariant subvarieties of V_5, including all $\mathfrak{A}_5$-orbits, a long list of low degree curves, a pencil of invariant anticanonical $K3$ surfaces, and a mildly singular surface of general type that is a degree-five cover of the diagonal Clebsch cubic surface. Furthermore, we discover two birational selfmaps of V_5 that commute with $\mathfrak{A}_5$-action and use them to describe the whole group of $\mathfrak{A}_5$-birational automorphisms. Finally, we prove our main result: the variety V_5 is $\mathfrak{A}_5$-birationally rigid, which means in particular that it cannot be $\mathfrak{A}_5$-equivariantly birationally transformed to a Fano threefold with mild singularities or a threefold fibered by rational curves or surfaces. As an application, we return to the starting point of our journey and produce three non-conjugate icosahedral subgroups in the Cremona group $\mathrm{Cr}_3(\mathbb{C})$, one of them arising from the threefold V_5.

One thing that we find really impressive is that all our classification results go hand in hand with each other, so that the purely birational objects like explicit birational selfmaps help to classify invariant curves, and they in turn help to deal with birational transformations.

The book has a clear motivational problem that is finally solved. While working on it, we discovered many relevant facts that are interesting on their own. Although some of them are not used in the proof of $\mathfrak{A}_5$-birational rigidity of V_5, we decided to include many of them, because we find them at least equally interesting as the initial birational question. We believe that these results can provide the same kind of aesthetic feeling as one that possibly stood behind the classical works of Klein, Maschke, Blichfeldt, and many others regarding symmetry groups. We hope that the reader will enjoy and appreciate them as well.

Acknowledgments

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Notation and conventions

Unless explicitly stated otherwise, all varieties are assumed to be projective and normal (but are allowed to be reducible). Everything is defined over the field $\mathbb{C}$ of complex numbers. Of course, the field $\mathbb{C}$ can be replaced by the reader's favorite algebraically closed field of zero characteristic.

By curves and surfaces we mean algebraic varieties of pure dimension 1 and 2, respectively. In particular, they can be reducible but are always reduced. By a conic we usually mean an irreducible plane curve of degree 2; if we need to allow a reducible curve, we try to indicate this explicitly.

An irreducible divisor is a divisor whose support is an irreducible variety; in particular, such a divisor is not necessarily prime. In many cases we do not make a distinction between divisors and classes of divisors. For example, given a Cartier divisor D on a variety X we may speak about D as an element of the Picard group of X. If X is a variety with a fixed embedding into $\mathbb{P}^n$, and D is a divisor on X that is cut out by a hypersurface in $\mathbb{P}^n$ of degree d, then we often refer to D simply as a hypersurface of degree d (or as quadric and cubic hypersurface for $d = 2$ and 3, respectively). By K_X we always denote the canonical class of X. Throughout the book we use the standard language of the singularities of pairs (see [30], [73], [74]).

Given two cycles Z_1 and Z_2 on a variety X, we denote their intersection cycle by $Z_1 \cdot Z_2$. If the latter is a zero-cycle, we will often use the same notation for its degree.

A complete linear system on a variety X is a projective space parameterizing all divisors that are linearly equivalent to a given (Weil) divisor D on X; we consider this notion only on normal varieties for simplicity. A linear system is a projective subspace of some complete linear system. The dimension of a linear system is the dimension of the corresponding projective space; in particular, pencils are linear systems of dimension 1. A mobile linear system is a non-empty linear system that does not have fixed components.

A projectivization $\mathbb{P}(V)$ of a vector space V is the projective space of all lines in V. A projectivization of a vector bundle is defined in a similar way. The Hirzebruch surface $\mathbb{F}_n$, $n \geqslant 0$, is defined as the projectivization of the vector bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$ on $\mathbb{P}^1$. In particular, one has $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{F}_1$ is the blow-up of $\mathbb{P}^2$ in one point.

Given a variety X and a sheaf $\mathcal{F}$ on X, we write $H^i(\mathcal{F})$ instead of $H^i(X, \mathcal{F})$ for brevity. The same applies to the notation $\chi(\mathcal{F})$ for the Euler characteristic of $\mathcal{F}$. We write $\chi_{top}(X)$ for the topological Euler characteristic of a variety X defined over $\mathbb{C}$. By $p_a(C)$ we denote the arithmetic genus $1 - \chi(\mathcal{O}_C)$ of a curve C.

A plane curve singularity given locally by an equation $x^2 = y^{n+1}$ for some $n \geqslant 1$ is called a *singularity of type* A_n. A *node* is a singularity of type A_1, i. e., one given locally by an equation $x^2 = y^2$. An *ordinary cusp* is a singularity of type A_2, i. e., one given locally by an equation $x^2 = y^3$. Also, we say that a variety X has a singularity of type A_n (or an ordinary cusp, respectively) along a smooth subvariety D of codimension 1, if the singularity locally in analytic topology looks like a product of D by a singularity of type A_n (or an ordinary cusp, respectively).

Let G be an algebraic group, and X be a variety with an action of G. We say that a subvariety of X is G-irreducible, if it is G-invariant and is not a union of two G-invariant subvarieties. If Y is a subvariety of X, we allow a small abuse of terminology and speak about a G-orbit of Y meaning the minimal G-invariant subvariety that contains Y. By a general point of Y we mean a point in a *dense* open subset in Y. This does not make much sense as an abstract notion, but is useful in some specific situations. In particular, a statement that a stabilizer of a general point of Y in G is trivial does make sense (it means the triviality of a stabilizer of a general point of every irreducible component of Y), although there is no concept of a stabilizer of a general point of Y. Similarly, we may say that a stabilizer of a general point of Y is isomorphic to some (fixed) group F, although there may be no way to identify stabilizers of points on different irreducible components of Y (which applies even to the case when Y is G-irreducible). Also, one can speak about a general point of Y being smooth. Suppose that Y is G-irreducible, its general point is smooth, and Z is a G-invariant subvariety, or a G-invariant divisor, or a G-invariant linear system on X. Then one can define the multiplicity of Z at a general point of Y as a local intersection index of Z with an appropriate number of general divisors from some very ample linear system passing through the corresponding point; we will denote the latter by $\mathrm{mult}_Y(Z)$. When we speak about a general curve on a variety without an action of a group, we mean a general curve of sufficiently large degree with respect to some ample divisor.

If $\mathcal{V}$ is a vector bundle on X, we say that $\mathcal{V}$ is G-invariant if for any element $g \in G$ the pull-back $g^*\mathcal{V}$ is isomorphic to $\mathcal{V}$; we say that $\mathcal{V}$ is a G-equivariant vector bundle when we have chosen a lifting of the action of G on X to $\mathcal{V}$.

If U is a representation of a finite group G, we denote by $U^\vee$ the dual representation of G. If X is a variety in $\mathbb{P}(U)$, we denote by $X^\vee$ the projectively dual variety in $\mathbb{P}(U^\vee)$.

By $\boldsymbol{\mu}_m$ we denote a cyclic group of order m. A dihedral group of order m is denoted by $\mathfrak{D}_m$. Note that $\mathfrak{D}_6 \cong \mathfrak{S}_3$ and $\mathfrak{D}_4 \cong \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$. A symmetric group of rank n is denoted by $\mathfrak{S}_n$, and an alternating group of rank n is denoted by $\mathfrak{A}_n$. To describe particular elements of the groups $\mathfrak{S}_n$ and $\mathfrak{A}_n$, we assume that these groups permute the numbers $1, \ldots, n$. Then we denote by $(i_1\, i_2 \ldots i_k)$ the cycle that sends i_1 to i_2 and so on, up to i_k that is sent to i_1. By $(i_{1,1} \ldots i_{k_1,1}) \ldots (i_{1,r} \ldots i_{k_r,r})$ we denote a composition of the corresponding cycles.

If Γ is some group, $G_1, \ldots, G_r$ are subgroups of Γ, and $g_1, \ldots, g_r$ are elements of Γ, then we denote by $\langle G_1, \ldots, G_s, g_1, \ldots, g_r\rangle$ the subgroup in Γ generated by $G_1, \ldots, G_s$ and $g_1, \ldots, g_r$.

Nearly every concept of algebraic geometry has its counterpart in the situation when all objects involved are acted on by some finite group G. If X is a variety with an action of G, we will sometimes say that X is a *G-variety*. A rational map $\phi\colon X \dashrightarrow X'$ between G-varieties X and X' is *G-equivariant*, if for each element g of the group G the diagram

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ X & \overset{\phi}{\dashrightarrow} & X' \end{array}$$

commutes. We say that ϕ is a *G-rational map* (or *G-map*), if there is an automorphism u of the group G such that for each element g of the group G the diagram

$$\begin{array}{ccc} X & \overset{\phi}{\dashrightarrow} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle u(g)} \\ X & \overset{\phi}{\dashrightarrow} & X' \end{array}$$

commutes. If a G-rational map ϕ is a birational map or a morphism, we sometimes say that ϕ is a G-birational map or a G-morphism, respectively. Similarly, if there exists a biregular G-morphism between two G-varieties, we will say that they are G-biregular. The reader should be aware that this terminology is not universally accepted, and in many sources a G-map means just a G-equivariant map. The maps that we call G-rational are referred to as rational maps of G-varieties in [35, §3].

Note that any G-equivariant rational map is a G-map. Contrary to this, if one takes two copies of a variety X with the same action of G, then a non-central element $g \in G$ defines a G-morphism $g\colon X \to X$ that is not G-equivariant. On the other hand, any element $g \in G$ defines a G-equivariant morphism $g\colon X \to X$, where the action on the second copy of X is constructed as the initial action twisted by an inner automorphism of G given by conjugation by g.

In general, if X' is a variety with an action of G, and u is an automorphism of G, one can produce another action of G on X' as the initial action twisted by u. Denote the variety X' with this twisted action by X'_u. Let X be some other variety with an action of G. We see that each G-rational map

$$\phi\colon X \dashrightarrow X'$$

gives rise to a G-equivariant rational map $\phi_u\colon X \dashrightarrow X'_u$ for a suitably chosen automorphism u of the group G. Since the actions of G on X' and X'_u have many properties in common (say, G has fixed points on X' if and only if it has fixed points on X'_u), it will sometimes be more convenient for us to work with G-equivariant maps, although in most cases our motivation will come from studying G-rational maps.

We denote by $\mathrm{Aut}(X)$ the group of (biregular) automorphisms of X, and by $\mathrm{Aut}^G(X)$ the group of G-automorphisms of X. If X is irreducible, we denote by $\mathrm{Bir}(X)$ the group of birational automorphisms of X, and by $\mathrm{Bir}^G(X)$ the group of G-birational automorphisms of X. We write $\mathrm{Cr}_n(\mathbb{C})$ for $\mathrm{Bir}(\mathbb{P}^n)$.

Since the largest part of the book is devoted to geometry of one particular variety, and we have to keep track of some objects for a long time, we naturally try to keep the same notation for such objects throughout the book. For example, we denote the quintic del Pezzo threefold by V_5 everywhere, denote the surface in V_5 that is the closure of the unique two-dimensional $\mathrm{PSL}_2(\mathbb{C})$-orbit in V_5 by $\mathscr{S}$, and denote the unique one-dimensional $\mathrm{PSL}_2(\mathbb{C})$-orbit in V_5 by $\mathscr{C}$ (which is one of the two $\mathfrak{A}_5$-invariant rational curves of degree 6 in V_5) starting from Chapter 7 up to the very end. On the other hand, in some special situations we may use one and the same symbol for two different objects in non-overlapping parts of the book, even when in one of the instances it is used for something basic for a long time; this happens when we feel that there is a similarity between the geometry of two varieties, which justifies similar notation for different objects that behave in a similar manner in these two situations. For example, we also use the symbol $\mathscr{C}$ to denote one of the two $\mathfrak{A}_5$-invariant rational curves of degree 6 in the Clebsch cubic surface. These similarities become most visible in Theorems 6.3.18 and 13.6.1 classifying low degree $\mathfrak{A}_5$-invariant curves in the Clebsch cubic surface and the quintic del Pezzo threefold, respectively. We tried to do our best not to cause any confusion for the reader with this convention.