7

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Introduction

This volume contains the proceedings of the Third Matscience Summer School held at Bangalore in September, 1966. The special feature of these proceedings was two systematic series of lectures, one by F. Pham of C.E.N., Saclay and CERN, Geneva and the other by G. Rickayzen of the University of Kent, Canterbury.

Pham dwelt at length on the applications of the methods of algebraic topology and differential forms to the study of the analytic properties of S-matrix theory, in particular, with reference to the location of singularities of the multiple scattering processes. This exposition was a natural sequel to the lectures of V. L. Teplitz, published in an earlier volume of this series.

Rickayzen discussed in detail the latest theory of superconductivity. Other lectures were those of Scadron, who dealt with some formal features of potential scattering theory, and B. M. Udgaonkar and A. N. Mitra, who spoke on certain aspects of bootstraps and quark models, respectively.

The contributions in pure mathematics in this volume include two lectures by S. K. Singh, one on the field of Mikusinski operators and another on Riemann mapping theorem, and a lecture on cosine functionals by P. L. Kannappan.

One of the highlights of the symposium was a lecture by S. K. Srinivasan who is keeping alive the interest of the Madras group in the theory of stochastic processes and who, in particular, has enlarged the domain of the application of the theory of product densities.

Alladi Ramakrishnan

Contents of Other Volumes

VOLUME 1

VOLUME 2

VOLUME 3

VOLUME 4

VOLUME 5

VOLUME 6

VOLUME 8

Contents

Superconductivity

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1. INTRODUCTION

I wish to present a review of our present understanding of the phenomenon of superconductivity. It was recognized in the early work of London and others that superconductivity is a cooperative effect of the electrons such that they condense into a many-body state described by a single wave function and this state is such that it resists deformations. The concept of such a cooperative effect is inherent in the wave function of Bardeen, Cooper, and Schrieffer[1] (referred to hereafter as BCS) where the Bloch states are paired, the different pairs being both occupied or unoccupied, in the configurations which are coherently superposed in the many-body wave function.

Landau[2] attempted to account for the rigidity of the wave function in terms of the low-lying excitations of the system. He was concerned with superfluidity but his analysis is easily carried over to superconductivity. If we suppose the system is uniform and isotropic, then in the ground state it will possess excitations with momentum $\mathbf{p}$ and energy $\epsilon(\mathbf{p})$. If the whole system is given a velocity $\mathbf{v}$, there will be a scattering mechanism which will tend to restore the system to equilibrium in the original rest frame. By Galilean invariance, in this frame the energy of the excitations is $\epsilon(\mathbf{p}) - \mathbf{p}\cdot\mathbf{v}$. If this is positive, electrons will remain in the condensate and superconductivity will be maintained. If it is negative, electrons will leave the condensate to become excitations and superconductivity will tend to be destroyed. Hence, Landau's criterion for superconductivity is that $\epsilon(\mathbf{p}) - \mathbf{p}\cdot\mathbf{v}$ should be positive for all $\mathbf{p}$. In the worst case of $\mathbf{p}$ parallel to $\mathbf{v}$, this requires

$$\frac{\epsilon(\mathbf{p})}{|\mathbf{p}|} > \mathbf{v}$$

In a superconductor $\mathbf{p} \approx \mathbf{p}_F$, the Fermi momentum. Hence, the criterion for superconductivity is that there should be no excitations of infinitely small energy, that is, there should be a gap in the spectrum of excitations. In the theory of BCS, there is indeed a gap and Landau's criterion is satisfied.†

It is now generally realized, however, that although Landau's criterion may be sufficient for superconductivity it is by no means necessary. Both theory and experiment now confirm that it is possible to have gapless superconductivity in, for example, thin films in a high parallel field[3] in a type-II superconductor when the field lies between H_{c_2} and H_{c_3}[4] and in a superconductor containing a sufficiently high concentration of paramagnetic impurities.[5] Even in a BCS-type superconductor at a finite temperature, Landau's criterion is broken because there are always phonons present to excite electrons from the condensate. In fact, this is strictly another case of gapless superconductivity.[6] Hence, in all the cases of practical importance the criterion is not valid. How is this possible? It is possible because, although the criterion shows when electrons will start to leave the condensate to form thermal excitations it does not show that all electrons will leave the condensate. In fact, electrons continue to leave the condensate until positive energy (or free energy) is required to create more. As long as some electrons remain in the condensate, superconductivity continues. Hence, Landau's criterion gives only the velocity at which some electrons leave the condensate, that is, it gives the velocity at which the density of superconducting electrons n_S, becomes less than the density of conduction electrons n. Since the criterion is broken in all the cases mentioned above, we expect $n_S = n$ only in a pure, homogeneous

†Strictly, one cannot apply the argument of Galilean invariance to a superconductor, because of the existence of the lattice. Most theoretical models of superconductivity including that of BCS, however, are Galilean invariant and lead to energy spectra of the form assumed above.

superconductor at the absolute zero. This conclusion is supported by many calculations. (Experiments usually do not measure n_S directly.)

If Landau's condition is not necessary for superconductivity, then what condition is? The answer is given directly in terms of the wave function or density matrix. In simplest terms it is that in a superconductor we have a new macroscopic variable $F(\mathbf{r})$ defined by

$$F(\mathbf{r}) = \langle \psi_{\downarrow}(\mathbf{r}) \psi_{\uparrow}(\mathbf{r}) \rangle$$

where $\psi_{\sigma}(\mathbf{r})$ is a field operator for electrons.† Here, the average is a quantum and thermodynamic average. If one also averages over small but macroscopic regions of space, one has that for a macroscopically uniform system in the absence of magnetic fields $F(\mathbf{r})$ has the form

$$F(\mathbf{r}) = F(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{r}}$$

The vector $\mathbf{p}$ represents the flow of the system, and the function $F(\mathbf{p})$ is determined by the condition that the system be in thermodynamic equilibrium subject to the condition that the phase of $F(\mathbf{r})$ be $\mathbf{p}\cdot\mathbf{r}$. In the BCS theory, a function with phase $(\mathbf{p}\cdot\mathbf{r})$ arises from the pairing of $(\mathbf{k} + \mathbf{p}/2, \uparrow)$ with $(-\mathbf{k} + \mathbf{p}/2, \downarrow)$.

The states corresponding to different values of $\mathbf{p}$ for which $F(\mathbf{p})$ is non-zero are such that the excitation energy of an electron in any one of them is positive and the matrix element of any single particle operator between any two of them is zero. Hence, one expects that scattering cannot reduce any one of these states to another; for each $\mathbf{p}$ we have a different state of metastable equilibrium. The condition, then, for superconductivity is that, for small $\mathbf{p}$, $F(\mathbf{p})$ should not be zero.

If this is true, one should be able to prove it. In fact, when impurity scattering is important, this has been shown by direct calculation for a number of different models.[5,8] Impurity scattering, however, is elastic and is, therefore, particularly inefficient for destroying superconductivity. One should show also that inelastic scattering such as scattering by phonons, does not destroy superconductivity. This has been done by considering the connection between

†This method assumes that the eigenstates do not have a definite number of particles present. An alternative method which allows eigenstates with definite numbers of particles has been given by Yang.[7]

the Meissner effect and the infinite conductivity.[9]† [More recently, this question has also been discussed by L. Pičman (see Ref. 14).] We propose to review this work here. Only the case of a simply connected superconductor will be considered since the problem of the persistent current in a multiply connected superconductor has been considered by Wentzel.[10]

2. MACROSCOPIC ANALYSIS

First, we consider what quantities we will have to calculate from the microscopic theory and what conditions they must satisfy in order that the Meissner effect and the infinite conductivity follow. Both these effects are weak field effects so we shall consider only the response of the system which is linear in the applied fields. Further, since both effects occur in transverse fields, we shall consider only the effects of such fields. Since the magnetic field and electric field are not independent in a general transverse field, we need consider only the effect of an electric field.

If the superconductor is also macroscopically uniform, different Fourier components will give rise to independent linear effects. The response of the system will be described by the induced current $\mathbf{j}_i(\mathbf{q}, \omega)$ and one component $\mathbf{E}(\mathbf{q}, \omega)$ will give rise to the corresponding current $\mathbf{j}_i(\mathbf{q}, \omega)$. Now, the only polar vector we can form from $\mathbf{q}$ and a transverse field $\mathbf{E}$ which is linear in $\mathbf{E}$ is $\mathbf{E}$ itself. Hence, the most general possible relation between $\mathbf{j}$ and $\mathbf{E}$ is

$$\frac{4\pi}{c^2}\,\mathbf{j}_i(\mathbf{q}, \omega) = \frac{iK(q, \omega)}{\omega}\,\mathbf{E}(\mathbf{q}, \omega) \tag{2}$$

where $K(q, \omega)$ is an arbitrary function of ω and of q, the modulus of the vector $\mathbf{q}$. The constants which appear in equation (2) are purely conventional.

The response of the system to a uniform field is given by equa-

†In a footnote to a recent preprint, P. C. Martin has argued that our considerations are not necessary to show that the Meissner effect and infinite conductivity go together. Once it is known that the system possesses a measureable conductivity in the experimentalist's sense, he argues, the two effects must imply each other. Martin's argument is very general but not obviously foolproof. If it should survive the test of time, then this paper confirms it and shows that according to the BCS theory a superconductor does possess a measureable conductivity.

tion (2) with $\mathbf{q}$ equal to zero. Hence, the frequency dependent conductivity is

$$\sigma(\omega) = \frac{ic^2}{4\pi\omega} K(0, \omega)$$

For infinite conductivity we require that this should tend to infinity as ω tends to zero. If we are more specific and ask for the electrons to behave at low frequencies like a gas of n_{S1} freely accelerating electrons then we require that as ω tends to zero,

$$\frac{\partial \mathbf{j}}{\partial t} = \frac{n_{S1}e^2}{m} \mathbf{E}$$

This means that as ω tends to zero,

$$K(0, \omega) \to \frac{4\pi n_{S1}e^2}{mc^2} \quad \text{a positive constant} \qquad (3)$$

To obtain the conditions for the Meissner effect we have to consider the effect of a static magnetic field on the superconductor. If the magnetic induction in the superconductor is $\mathbf{B}$ we have, from Maxwell's equations, that for a general field

$$i\mathbf{q} \times \mathbf{E} = \frac{i\omega\mathbf{B}}{c}$$

If the field is transverse, this can be rewritten

$$\mathbf{E} = -\frac{\omega\mathbf{q} \times \mathbf{B}}{cq^2}$$

and the relation between induced current and magnetic field is

$$\frac{4\pi}{c} \mathbf{j}_i(q, \omega) = -\frac{iK(q, \omega)}{q^2} (\mathbf{q} \times \mathbf{B})$$

In the limit of static fields, this is

$$\frac{4\pi}{c} \mathbf{j}_i(\mathbf{q}, 0) = -\frac{iK(\mathbf{q}, 0)}{q^2} (\mathbf{q} \times \mathbf{B})$$

If a source $\mathbf{j}_S$ is present, we have that $\mathbf{B}$ is determined by

$$\text{curl}\, \mathbf{B} = \frac{4\pi}{c} (\mathbf{j}_i + \mathbf{j}_S)$$

For a given source $\mathbf{j}_S$, these equations can be solved for the induction $\mathbf{B}$. For the special case of a long, thin, superconductor parallel to a uniform external field H with specular reflection of the electrons at the boundary, the solution is easily obtained.[11] The induction at a distance z from the surface is

$$B(z) = -\frac{iH}{\pi}\int_{-\infty}^{+\infty} dq \frac{qe^{iqz}}{q^2 + K(q, 0)}$$

The behavior of B at large distances from the surface is determined by the behavior of the integrand for small q. If we require that the field decrease exponentially with distance at large distances from the surface, then we must have

$$\lim_{q\to 0} K(q, 0) = \text{positive constant}$$

By analogy with equation (3), we shall write the constant as $4\pi n_S e^2/mc^2$. Hence, for a Meissner effect in the above sense,

$$\lim_{q\to 0} K(q, 0) = \frac{4\pi n_{S2} e^2}{mc^2} \tag{4}$$

We see then that the two effects depend on the behavior of $K(q, \omega)$ as q and ω tend to zero, the only difference being in the order in which the limits are taken. One might surmise that the order of the limits is irrelevant and that the two quantities (3) and (4) are always equal. This would correspond with the idea of Section (1) that the current carrying states are in metastable equilibrium. Nevertheless, it is possible to construct a trivial example where the limits are not equal. This is the case of a free electron gas for which all the electrons are freely accelerate and so n_{S_1} is equal to n, whereas there is no Meissner effect and n_{S_2} is zero. However, this case includes no scattering to bring the system into equilibrium with its surroundings. We might guess, therefore, that when such scattering is present, the two limits are equal and the phenomena are equivalent. This is the approach of Ref. 9 which we review here.

3. FORMALITIES

We have shown that the effects depend on the one function $K(q, \omega)$ which, to determine whether or not the effects exists, must be calculated from the appropriate microscopic theory. We shall delay introducing a specific theory for as long as possible, deriving instead formal expressions for $K(q, \omega)$ in terms of microscopic quantities. In this way we shall be able to pinpoint the condition necessary for the two phenomena to occur together.