Lecture Notes in Mathematics

Edited by A. Dold and B. Eckmann

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AMS Subject Classifications (1970): XX-46

ISBN 3-540-08441-X Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-08441-X Springer-Verlag New York Heidelberg Berlin

Printed in Germany

Printing and binding: Beltz Offsetdruck, Hemsbach/Bergstr.
2141/3140-543210

Introduction

In 1972, Alfsen and Effros ([AE]) published a paper in which they discussed certain problems concerning the isometrical structure of Banach spaces. Using this paper as a starting point, a research group at the Freie Universität Berlin has been working on structural problems of this type since the spring of 1973. During this period the results achieved by the group have been published as papers, theses, and preprints. The purpose of these notes is to give the reader a more or less complete account of those results which can be grouped under the heading "L^p-structure".

Let X be a real Banach space and $1 \leq p \leq \infty$. Two closed subspaces J, $J^\perp$ of X are called complementary L^p-summands if X is the algebraic sum of J and $J^\perp$ and for every $x \in J$, $x^\perp \in J^\perp$

$$\|x + x^\perp\|^p = \|x\|^p + \|x^\perp\|^p \quad (\text{if } 1 \leq p < \infty)$$

$$\|x + x^\perp\| = \max\{\|x\|, \|x^\perp\|\} \quad (\text{if } p = \infty),$$

i. e. when the elements in J and $J^\perp$ behave like disjoint elements in an L^p-space. The projection from X onto J corresponding to this decomposition of X is called an L^p-projection and the set of all projections obtained in this way $\mathbb{P}_p$.

L^1- and L^∞-summands and the corresponding projections were first studied by Cunningham ([C1] and [C2]). Alfsen and Effros carried on the investigation in the above-mentioned paper, in which probably the most important results are the characterization of M-ideals by means of an intersection property (an M-ideal is a closed subspace whose polar is an L^1-summand in the dual space) and the introduction of the structure topology, with whose help one can prove a very generelized form of the Dauns-Hofmann theorem. They also applied the concepts to the most important concrete Banach spaces.

The general case, i. e. p not necessarily =1 or = ∞, has hardly been investigated at all, apparently. The main reason is that even very simple questions (e. g. Is the intersection of two L^p-summands also an L^p-summand) can only be answered if it is known that every pair of L^p-projection commute . In the case of p = 1 or p = ∞ this is easily seen and can also be proved directly for Banach spaces where the Clarkson inequality (cf. [L] , p. 169) is valid for the relevant p. Behrends has shown in [B2] that the answer to this question is affirmative for all $p \neq 2$. Since any orthogonal projection in a Hilbert space is an L^2-projection this result does not hold for p=2. On the other hand some important results hold for any complete Boolean algebra of L^p-projections, not necessarily containing them all. In chapters 3 - 5 we therefore formulate these results in the general context, which in particular means that we can apply them to the case p = 2 by considering maximal families of commuting projections.

Some authors ([CS],[E1]) have studied a natural generalization of the concept of L^p-summand. Let F be a mapping from $R_+ x R_+$ into R_+. We call two subspaces J, $J^\perp$ of a Banach space X, F-summands, if X is the algebraic direct sum of J and $J^\perp$ and further $F(\|x\|,\|x^\perp\|) =$

$\|x + x^{\perp}\|$ for all x in J, $x^{\perp}$ in $J^{\perp}$. ([CS] considers the special case $F(s,t) = f^{-1}(f(s) + f(t))$ for a continuous strictly monotone function f: $\mathbb{R}_{+} \to \mathbb{R}_{+}$.) It can be shown (see note at the end of chapter 1) that if there are two nontrivial F-summands, one contained in the other, then $F = F_p$ for some p in $[1,\infty]$ whereby $F_p(s,t) = (s^p+t^p)^{1/p}$ for $p<\infty$ and $F_\infty(s,t) = \max\{s,t\}$. In this case F-summands are of course L^p-summands so that a restriction of our consideration to the latter does not involve any real loss of generality.

The main problem for choosing the material for these notes was that, while in the case of $p \neq 1, 2, \infty$, the definitions, propositions, proofs etc. are formally identical (differing only in the value of p) for all p, in the case $p = 1$ and $p = \infty$ the propositions are often only valid in a modified form or cannot be proved by the same method as in the general case. In the interest of uniformity we have therefore only mentioned those results which can be proved in more or less the same way as in the general case. (some results which we have left out for this reason can be found in [DGM]). The difference in the behaviour of the case $p = 1$ and $p = \infty$ as opposed to the other values of p is basically due to the fact that L^p-projections in dual spaces are necessarily w^*-continuous for $p > 1$ but not for $p = 1$ (which in particular means that M-ideals are not necessarily L^∞-summands). This result was proved independently by [F] and [E1] (in [E1] in a more general form for dual F-projections) although the method of the proof is the same as in [CER], who only consider the case $p = \infty$.

These notes fall into two main parts - chapters 1-3 in which the theory is developed and chapters 4-7 which deal with some applications. The contents of the individual chapters are as follows:

Chapter 1: The concept of an L^p-summand is explained with the help of some concrete examples. Although the proof of a lemma concerning the effect of transposition to L^p-summands and -projections is given in full, the main theorem concerning the commutativity of L^p-projections is only stated. A sketched proof can be found in appendix 1. It is shown that, for $p \neq 2$, $\mathbb{P}_p$ is a Boolean algebra and, for $p < \infty$, a complete one in which increasing nets converge pointwise to their suprema.

Chapter 2: The Cunningham p-algebra $C_p(X)$ (closure of the linear hull of $\mathbb{P}_p$ in $[X]$) and the Stonean space Ω of $\mathbb{P}_p$ (whose clopen subsets represent $\mathbb{P}_p$) are defined and examined. In particular it is shown that the Cunningham p-algebra is isomorphic in all structures to the space of continuous functions on Ω. The effect of taking products and quotients is also investigated.

Chapter 3: In the first part of the chapter we show how a Banach space X can be embedded in a field of Banach spaces over Ω in such a way that the L^p-projections in X have the effect of characteristic projections. This embedding (p-integral module representation) turns out to be the most important aid in the investigation of L^p-structure.

The second part contains some important consequences which are needed in the following chapters.

Chapter 4: With the help of the techniques of chapter 3 we show that abstract L^p-spaces can be characterized by the maximality of their L^p-structure - a Banach space X is isometric to an L^p-space if and only if $(C_p(X))_{COMM} = C_p(X)$. The most important result used in the proof is a lemma concerning the existence of projections in $(C_p(X))_{COMM}$ which generalizes a result of Cohen-Sullivan ([CS]) for smooth reflexive spaces.

We also give an explicit description of the L^p-summands in an L^p-space. It turns out that every L^p-summand is more or less the annihilator of a measurable set, a result already obtained in [Su 2].

Chapter 5: We study the relationship between the p-integral representation of a Banach space and the p'-integral representation of the dual ($1/p + 1/p' = 1$), in particular the connection between the reflexivity of the space itself and that of the component spaces in the representation.

Chapter 6: In an analogous manner to the theory of self-adjoint operators in Hilbert space we represent the operators in the Cunningham p-algebra as Stieltjes integrals over spectral families of projections. It is shown that there is a 1-1 correspondence between the operators in $C_p(X)$ and normalized spectral families. We then give some important results in the general theory which follow from this.

Chapter 7: In this chapter we apply the representation of chapter 3 to some simple vector-valued L^p-spaces and draw some parallels to the general case.

In chapter 0 we have collected those results from other branches of mathematics which the reader will need to understand the following chapters. The appendices contain a sketched proof of theorem 1.3 (for a complete proof see [B2]), some remarks concerning the structure of the L^{∞}-summands in CK-spaces, and a discussion of a measure theoretic approach to integral modules.

It is clear that when a group have been working together for several years it is impossible to say which member is responsible for each result. Without forgetting this we can say however that the contributions of the individual members of the group are roughly as follows:- Chapter 1: Behrends ; Chapter 2: Danckwerts, S. Göbel, Meyfarth ; Chapter 3: Evans (section F together with Greim) ; Chapter 4: Evans ; Chapter 5: Greim ; Chapter 6 : Müller ; Chapter 7: Evans, Greim.

In conclusion we would like to thank the FNK (Kommission für Forschung und wissenschaftlichen Nachwuchs) of the Freie Universität Berlin for assisting us financially in the years 1974-75.

C O N T E N T S

Chapter 0: Preliminaries

Topology

We assume that the reader is familiar with the elementary concepts of topology.

A nowhere dense set is a set A in a topological space such that the interior of the closure of A is empty. A set which is the union of countably many nowhere dense sets is said to be of first category. In a compact space ("compact" always includes the Hausdorff property) the empty set is the only open set of first category.

A topological space is said to be extremally disconnected when the closure of each open set is also open. If the space is also Hausdorff this implies that the connected components consist only of single points or in other words that the space is totally disconnected.

Consider the collection of sets of the form $A \Delta B$ where A is a clopen set and B is a set of first category. This collection is clearly closed under finite unions and intersections. If the space is extremally disconnected it is also closed under countable unions since the union of countably many sets of first category is also of first category and the union of countably many clopen sets is open and thus, since its closure is clopen differs from a clopen set by a set of first category. Since the complement of such a set also has this form it follows that in an extremally disconnected space the sets of this form form a σ-algebra. This σ-algebra contains the open sets since the closure of an open set is clopen and the boundary is of first category. In a compact space the equality $A \triangle B = C \triangle D$ (A, C clopen, B,D of first category) implies that $A = C$ and $B = D$ since $\emptyset$ is the only clopen set of first category.

A subset A of a topological space is called regularly closed if the closure of the interior of A is A. Since the interior of A is open it follows that in an extremally disconnected space the regularly closed sets are the clopen sets. In an extremally disconnected compact space the closure of an open set is homeomorphic to its Stone-Čech-compactification ([Sch], II.7.1).

Borel measures

In a topological space the Borel sets are the members of the σ-algebra generated by the open sets. It follows from the first part of this chapter that the Borel sets in a compact extremally disconnected space can be uniquely represented as the difference of a clopen set and a set of first category. A Borel measure is a σ-additive set function defined on the Borel sets. The support of a Borel measure m, denoted by supp m, is the set of all points such that every neighbourhood of the point contains a set with non-zero measure. The support is always closed. A Borel measure is said to be regular (from inside) if the measure of each set is the limit of the measures of the compact sets contained in it. The support of a regular Borel measure is regularly closed. The Riesz representation theorem states that the space of all finite regular Borel measures on a compact space is the dual of the space of continuous functions on this space (with the sup-norm) under the duality $\langle\mu, f\rangle := \int f d\mu$. A regular <u>content</u> is a set function defined on the compact subsets with the following properties:

(i) $0 \le m(D) < \infty$

(ii) $C \subset D$ implies $m(C) \le m(D)$

(iii) $m(C \cup D) \le m(C) + m(D)$

(iv) $m(C \cup D) = m(C) + m(D)$ for disjoint C, D

(v) $m(D) = \inf\{m(C) \mid D \subset C^{o}\}$

In a compact space every regular content can be extended to an unique regular Borel measure ([H2], §§53,54).

If m and m' are two finite Borel measures on a topological space and every set with zero m-measure has also m'-measure zero the Radon-Nikodym theorem states that there is an m-integrable function f such that m' = fm. The theorem can clearly be extended to apply to measures which are constructed from finite measures with pairwise disjoint support.

<u>Boolean algebras</u>

A Boolean algebra is a distributive complemented lattice with maximal and minimal element. With each Boolean algebra $\mathfrak{A}$ we associate a compact totally disconnected topological space Ω in the following manner. We consider the trivial Boolean algebra $2 := \{0,1\}$ as a topological space with the discrete topology and define Ω as the set of all homomorphisms of Boolean algebras from $\mathfrak{A}$ in 2. Thus Ω is a closed subspace of the compact totally disconnected space $2^{\mathfrak{A}}$ and so also compact and totally disconnected. Ω is called the

Stonean space of the Boolean algebra $\mathfrak{A}$. The mapping $a \mapsto B_a :=$ $\{ f \mid f \in \Omega, f(a) = 1 \}$ is an isomorphism of Boolean algebras from $\mathfrak{A}$ to the Boolean algebra of clopen subsets of Ω. (See e. g. [H1]).

A Boolean algebra in which every subset has an infimum and a supremum is called complete. A Boolean algebra is complete if and only if its Stonean space is extremally disconnected.

Every Boolean algebra is ordered in a natural way by the order $a \leq b \Leftrightarrow a \wedge b = a$. If $(\mathfrak{A}_i)_{i \in I}$ is a family of Boolean algebras the cartesian product of the $\mathfrak{A}_i$'s can be made into a Boolean algebra by defining the lattice operations component-wise. This Boolean algebra is called the product of the Boolean algebras $\mathfrak{A}_i$ and is written $\prod_{i \in I} \mathfrak{A}_i$.

Chapter 1: L^p-projections

X is always a Banach space over the reals. We define certain subspaces of X and investigate some properties which they have. These subspaces will be considered in much more detail in the following chapters.

1.1 Definition: Let $1 \leq p \leq \infty$, $J \subset X$ a subspace, $E : X \to X$ a projection (that is E linear, $E^2 = E$).

(i) J is called L^p-summand, if there is a subspace $J^\perp$ such that algebraically $X = J \oplus J^\perp$, and for $x \in J$, $x^\perp \in J^\perp$ we always have $\|x+x^\perp\|^p = \|x\|^p + \|x^\perp\|^p$ (if $p = \infty$: $\|x+x^\perp\| = \max \{\|x\|, \|x^\perp\|\}$).

(ii) E is called L^p-projection, if for every $x \in X$

$\|x\|^p = \|Ex\|^p + \|x-Ex\|^p$ (if $p = \infty$: $\|x\| = \max \{\|Ex\|, \|x-Ex\|\}$)

1.2 Proposition:

(i) For any L^p-summand J the subspace $J^\perp$ in definition 1.1(i) is uniquely determined. We therefore call $J^\perp$ "the L^p-summand complementary to J" and write $X = J \oplus_p J^\perp$.

(ii) Let J be an L^p-summand and E be the projection onto J with respect to $X = J \oplus_p J^\perp$. Then E is an L^p-projection.

(iii) For any L^p-projection E the spaces range E and ker E are complementary L^p-summands, that is $X = \text{range } E \oplus_p \ker E$.

(iv) Every L^p-projection E is continuous with $\|E\| \leq 1$.
In particular, L^p-summands J are closed (since $J = \ker E^\perp$, where $E^\perp$ is the L^p-projection onto $J^\perp$).

(v) There is a one-to-one correspondence between the set of

L^p-summands and the set of L^p-projections.

Proof:

(i) Let J be an L^p-summand, such that $J_1^\perp$ and $J_2^\perp$ satisfy the conditions of definition 1.1(i). We will prove $J_1^\perp = J_2^\perp$.

Let $y \in J_1^\perp$. We have $y = x + x^\perp$ where $x \in J$, $x^\perp \in J_2^\perp$. For $p < \infty$ it follows that $\|y\|^p = \|x\|^p + \|x^\perp\|^p$. On the other hand, $\|x^\perp\|^p = \|x\|^p + \|y\|^p$ (because $x^\perp = -x + y$, $x \in J$, $y \in J_1^\perp$), hence $x = 0$ and $y = x^\perp \in J_2^\perp$.

If $p = \infty$, consider $y + ax$ ($= (a+1)x + x^\perp$) for $a > 0$. Condition 1.1(i) implies $\max\{\|(a+1)x\|, \|x^\perp\|\} = \|y+ax\| = \max\{\|ax\|, \|y\|\}$, so necessarily $x = 0$.

We have thus proved that $J_1^\perp \subset J_2^\perp$. The reverse inclusion follows by symmetry.

(ii), (iii), (iv), (v) are easily verified. □

Examples:

1) Let $1 \leqq p \leqq \infty$ and (S, Σ, μ) a measure space. In $X = L^p(S, \Sigma, \mu)$, every measurable subset $B \subset S$ defines an L^p-projection by $f \mapsto f\chi_B$. The measurability of B is not essential. It is sufficient that for $f \in X$ always $f\chi_B \in X$ (that means $B \cap D \in \Sigma$ for $D \in \Sigma$, $\mu(D) < \infty$). We investigate the structure of the L^p-projections on X in more detail in chapter 4.

2) Every closed subspace J of a Hilbert space is an L^2-summand. $J^\perp$ is the usual space orthogonal to J, and the norm condition is the Pythagorean law for orthogonal elements.

3) Let T be a topological space and $S \subset T$ a clopen subset. The annihilator of S, $\{f \mid f : T \to \mathbb{R}$ continuous and bounded, $f|_S = 0\}$ is an L^∞-summand in the space of all real-valued continuous and bounded functions on T. We will show in appendix 2 that for compact T all L^∞-summands have this form.

4) The operators Id and 0 are always L^p-projections. We say that <u>the L^p-structure of X is trivial</u> if there are no other L^p-projections (or equivalently: there are no other L^p-summands than X and $\{0\}$).

5) If X and Y are Banach spaces, $1 \leq p \leq \infty$, define the norm on $X \times Y$ by $\|(x,y)\| := (\|x\|^p + \|y\|^p)^{1/p}$ (if $p = \infty$: $\|(x,y)\| = \max\{\|x\|, \|y\|\}$). As subspaces of $X \times Y$, X and Y are complementary L^p-summands, and up to isometric isomorphism L^p-summands always have this form.

We now state a theorem concerning L^p-projections which is fundamental to the following investigations. Motivated by results of [C] and [AE] (L^1- and L^∞-projections there are called L- and M-projections, respectively) it would seem reasonable to attempt to prove a commutativity theorem for L^p-projections which seemed to be essential for nearly all results. A thorough study of certain classes of Banach spaces (CK-spaces, AK-spaces, L^p-spaces; cf. [B1] and [Sü 2]) showed that in these classes L^p-projections always commute if $p \neq 2$, and every Banach space admits nontrivial L^p-projections for at most one p in $[1,\infty]$. Of course, L^2-projections will not commute in general,