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Josip Plemelj

PROBLEMS IN THE SENSE OF RIEMANN AND KLEIN

Edited and translated by J. R. M. Radok

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FOREWORD

IT IS a privilege indeed to welcome a book by Professor Plemelj on a subject in the development of which he played an historic role. Professor Plemelj's place in the history of mathematics is, of course, secure, but we note with particular pleasure that at his advanced age he is again enriching the mathematical literature.

The chapters of this book converge to the solution of two central classical problems which were, and still are, the source of numerous mathematical results of considerable importance in theoretical and physical applications. Klein proposed and stimulated interest in one phase of the first problem which is concerned with the mapping properties of the ratio of two linearly independent solutions of a Fuchsian differential equation. Riemann originated the second problem which has to do with the determination of n functions, each analytic in a domain D, and such that the limit value of each function at the interior side of the closed boundary C is equal to a linear combination of the n exterior limit values of functions required to be analytic in the exterior of C. The solution of this problem leads to the explicit solution of certain fundamental integral equations with Cauchy kernels; and it subsumes, among other things, the solution of equations of Wiener–Hopf type.

Professor Plemelj has long been interested in these fascinating problems of Klein and Riemann; and some of his own original contributions are contained in the text. His account of methods for solving the problems is of more than historical value. He shows that the problems are still attractive by discussing recent work and pointing to phases of the problems which remain to be solved.

In Part I of the text Professor Plemelj develops the theory of ordinary linear differential equations with analytic coefficients. After a presentation of the general theorems about these equations, there is a detailed analysis of the hypergeometric equation; the second order

Fuchsian differential equation; and the singular points of these equations. This is followed by a discussion of Euler's method for finding definite integral representations for the solutions of differential equations and it is shown how this kind of representation gives the global behaviour of the solutions. The theory in this part is directed toward Chapters 7 and 8 which include the solution of some of Klein's problems and a presentation of some of Klein's theorems about mapping with the ratio of two independent solutions of a Fuchsian equation. There is a discussion of some of Plemelj's own work and its relation to some recent advances. Plemelj shows that Klein's problem is still alive and his concluding remarks in Chapter 10 suggest certain interesting but unanswered questions.

Part II begins with a review of Fredholm's integral equation and the method Fredholm used to solve it. The usual Fredholm theorems are proved and enunciated. The results are then applied to Dirichlet's problem for the potential equation. Chapter 14 is devoted to an explanation of the behaviour of the limit values of analytic functions expressed as Cauchy integrals. The formulae which are given are very useful for finding the solution of certain singular integral equations; and they are now known as Plemelj's formulae. The goal of the development in this part is the solution of Riemann's problem. This is given in considerable detail in Chapter 15. The theory of this chapter is of current interest and is being pushed forward because it is intimately connected with the theory of integral equations and other function theoretic problems.

The reader of this book will find no exercises, and there are only a few subsidiary problems discussed in the text. Problems which are suggested by the reader's own imagination and interest are tacitly left for him to solve. This is so because Professor Plemelj's purpose is to present the more or less mature student with an opportunity to pursue a train of thought which runs, with a minimum of side excursions, to a comprehension of the famous problems of Klein and Riemann.

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PART I

LINEAR DIFFERENTIAL EQUATIONS

Linear differential equations represent a very important type of differential equations, not only because many purely mathematical problems, but also, because by far the majority of all questions of practical applications lead to such differential equations. In function theory, physics and astronomy, we find a large number of examples of linear differential equations of the ordinary as well as of the partial type. Problems arising in the theory of small vibrations are formulated mathematically by use of linear differential equations. It is also this type of differential equations to which the mathematicians of the last century have devoted their greatest efforts and on which they have largely expanded the function theoretical tools, so that one might very well claim that today, mathematics controls this kind of differential equations to quite a large extent. Originally, the problem of the solution of a differential equation was conceived to be its reduction to integrals where the upper limit was expressed in terms of the independent variable, i.e. to so-called quadratures. However, it then turned out that a solution in this form was rarely possible, and hence that the term 'integration of a differential equation' was unsuitable. It became necessary to reformulate the problem in the form: the solution has to be examined by use of the differential equation itself and one has to deduce from the differential equation directly the course and the properties of its solution. This formulation is more general than the original one: it has to be recognized on the basis of the differential equation itself what the properties of its solution are. This is especially important in function theory. In the most frequent case when the coefficients are analytic and we have a solution, for example, in the form of a power series, we have to deduce from the differential equation the behaviour of the solution in the entire region where it exists, i.e. for every analytic continuation of the independent variable. This problem has been solved for ordinary linear differential equations. This type of linear differential equations will be treated first.

CHAPTER 1

Ordinary Differential Equations

§ **1.** The linear, nth order differential equation in the independent variable x and the dependent variable y has the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = b, \quad (1.1)$$

where the coefficients $a_1, a_2, \ldots, a_n$ of the differential equation as well as b depend on x and will be assumed to be known.

For $b = 0$, the differential equation is called homogeneous, otherwise non-homogeneous. The differential equation can be reduced to a system of differential equations of the form

$$\begin{aligned} \frac{dY_1}{dx} &= a_{11} Y_1 + a_{12} Y_2 + \cdots + a_{1n} Y_n + b_1, \\ \frac{dY_2}{dx} &= a_{21} Y_1 + a_{22} Y_2 + \cdots + a_{2n} Y_n + b_2, \\ &\cdots\cdots\cdots\cdots \\ \frac{dY_n}{dx} &= a_{n1} Y_1 + a_{n2} Y_2 + \cdots + a_{nn} Y_n + b_n, \end{aligned} \quad (1.2)$$

where y and the derivatives dy/dx, $d^2y/dx^2, \ldots, dy^{n-1}/dx^{n-1}$ have been introduced as new dependent variables $Y_1, Y_2, \ldots, Y_n$. Conversely, the system of linear differential equations can be reduced to a single linear differential equation.

Following Liouville, the solution of the system of linear differential equations (1.2) is readily found by reducing it after multiplication of each equation of (1.2) by dx and subsequent integration to a system of linear integral equations

$$Y_1(x) = Y_1(x_0) + \int_{x_0}^{x} [a_{11}Y_1(\xi) + a_{12}Y_2(\xi) + \cdots + a_{1n}Y_n(\xi) + b_1]\,d\xi,$$

$$Y_2(x) = Y_2(x_0) + \int_{x_0}^{x} [a_{21}Y_1(\xi) + a_{22}Y_2(\xi) + \cdots + a_{2n}Y_n(\xi) + b_2]\,d\xi,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$Y_n(x) = Y_n(x_0) + \int_{x_0}^{x} [a_{n1}Y_1(\xi) + a_{n2}Y_2(\xi) + \cdots + a_{nn}Y_n(\xi) + b_n]\,d\xi. \tag{1.3}$$

Substituting successively for $Y_1, Y_2, \ldots, Y_n$ under the integral signs on the right-hand sides of these equations, one obtains for the $Y_k(x)$ series a series the convergence of which is easily compared with the convergence of a series of the form

$$g + gKs + g\frac{K^2s^2}{1.2} + g\frac{K^3s^3}{1.2.3} + \cdots,$$

where g and K are positive numbers, while s represents the length of the path of integration. This solution exists for arbitrary initial values $Y_1(x_0), Y_2(x_0), \ldots, Y_n(x_0)$ for every x_0 for which all the coefficients a_{kh} and b_k remain finite. Thus, one can already deduce from the coefficients the values of x_0 for which the solution could be in doubt. If the coefficients are analytic, the solutions will also be analytic and the successive integrations lead only to analytic functions. For non-analytic functions, it is necessary to specify the path of integration, i.e. it is not immaterial by which path one travels from the starting point x_0 to the end point x.

It is important by which course we pursue the continuous history of x from the initial point x_0 to the point x. The existence of a unique solution is ensured only if the path lies throughout in the domain of regularity of the coefficients a_{kh} and b_k. All the integrals must be evaluated along the same path. Where then can we have singularities of the solution? Only where the coefficients a_{kh} and b_k are singular. However, even the singular behaviour of the coefficients is not always a hurdle to the regular behaviour of the solution; nevertheless, wherever the coefficients $a_{kh}(x)$ and $b_k(x)$ are regular, the solutions are certainly also regular.

So far we have spoken about the solutions of the system of differential equations each of which is of the first order. It is not difficult to extend the existence theorem to an nth order linear differential equation. In that case, the existence theorem states that we can prescribe for the solution y of an nth order linear differential equation at the initial point x_0 the values $y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0)$ in a completely arbitrary manner. One, and only one, solution $y(x)$ will exist which is regular along a given path from x_0 to x, if only the entire path lies in a region where all coefficients are regular.

§ **2.** We will now restrict our considerations to a homogeneous nth order linear differential equation. We will soon see that this restriction is admissible, because we can always reduce the solution of the non-homogeneous linear differential equation to a solution of the homogeneous equation and several quadratures.

First of all, we show that the general solution of a homogeneous linear differential equation can be constructed as soon as suitable particular solutions are known. We note the following. If $y_1, y_2, \ldots, y_k$ are solutions, then also every expression of the form $c_1y_1 + c_2y_2 + \cdots + c_ky_k$ is again a solution, where $c_1, c_2, \ldots, c_k$ are arbitrary constants.

One says that this solution can be represented *linearly* in terms of the solutions $y_1, y_2, \ldots, y_k$. In this context, the concept of the linear dependence of functions $y_1, y_2, \ldots, y_k$ suggests itself. Functions $y_1, y_2, \ldots, y_k$ are said to be linearly dependent on each other, if there is a system of k constants $c_1, c_2, \ldots, c_k$, not all of which are identically equal to zero, such that the equation $c_1y_1 + c_2y_2 + \cdots + c_ky_k = 0$ holds for arbitrary x in the domain where the functions $y_1, y_2, \ldots, y_k$ are defined. If there is no such system of constants, then the functions $y_1, y_2, \ldots, y_k$ are said to be *linearly independent* of each other.

Now we form n solutions $y_1, y_2, \ldots, y_n$ of the linear differential equation which are linearly independent of each other by prescribing their values $y_k(x_0), y_k'(x_0), y_k''(x_0), \ldots, y_k^{(n-1)}(x_0)$, $k = 1, 2, \ldots, n$ at the point $x = x_0$ in the following manner:

$$\begin{aligned} y_1(x)&: \quad 1, 0, 0, \ldots, 0, \\ y_2(x)&: \quad 0, 1, 0, \ldots, 0, \\ y_3(x)&: \quad 0, 0, 1, \ldots, 0, \\ &\ldots\ldots\ldots\ldots, \\ y_n(x)&: \quad 0, 0, 0, \ldots, 1. \end{aligned} \tag{1.4}$$

The existence theorem tells us that this is possible. Every other solution can be represented linearly in terms of these solutions. In fact, if we select for any solution $y(x)$ as initial values of $y(x_0)$, $y'(x_0)$, $y''(x_0), \ldots, y^{(n-1)}(x_0)$, the constants $c_1, c_2, \ldots, c_n$, then there exists a single such solution which is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + c_3 y_3(x) + \cdots + c_n y_n(x).$$

On the basis of (1.4), it is immediately seen that $y(x_0) = c_1$, and that the successive derivatives have the values $y'(x_0) = c_2$, $y''(x_0) = c_3, \ldots$, $y^{(n-1)}(x_0) = c_n$. The functions $y_1(x), y_2(x), \ldots, y_n(x)$ constructed above are linearly independent. In fact, if we assume that there is between them a relation

$$k_1 y_1(x) + k_2 y_2(x) + \cdots + k_n y_n(x) = 0$$

with constant coefficients $k_1, k_2, \ldots, k_n$ which is valid for every x, then we could differentiate successively this equation with respect to x and would find for $x = x_0$ consecutively $k_1 = 0, k_2 = 0, \ldots, k_n = 0$, i.e. that the functions $y_1(x), y_2(x), \ldots, y_n(x)$ are linearly independent.

§ 3. We will now pose a general question regarding a process by which one can recognize whether m given functions $y_1(x), y_2(x), \ldots$, $y_m(x)$ are linearly dependent on each other. In that case it should be possible to find m constants $c_1, c_2, \ldots, c_m$ not all of which are zero such that for arbitrary x one has the equations

$$\begin{aligned} c_1 y_1(x) + c_2 y_2(x) + \cdots + c_m y_m(x) &= 0, \\ c_1 y_1'(x) + c_2 y_2'(x) + \cdots + c_m y_m'(x) &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ c_1 y_1^{(m-1)}(x) + c_2 y_2^{(m-1)}(x) + \cdots + c_m y^{(m-1)}(x) &= 0. \end{aligned} \tag{1.5}$$

All these equations are obtained from the first equation by successive differentiations.

These m homogeneous equations which are linear in the $c_1, c_2, \ldots, c_m$ series should have a non-trivial solution, i.e. they should yield a system of constants $c_1, c_2, \ldots, c_m$ not all of which are zero. In order that this will be possible, it is only necessary that the determinant

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_m(x) \\ y_1'(x) & y_2'(x) & \ldots & y_m'(x) \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(m-1)}(x) & y_2^{(m-1)}(x) & \ldots & y_m^{(m-1)}(x) \end{vmatrix} \tag{1.6}$$

is identically equal to zero for every value of x. This determinant is called the *Wronskian* of the functions $y_1, y_2, \ldots, y_m$. We have, of course, assumed here that all of the functions $y_1(x), y_2(x), \ldots, y_m(x)$ have $m - 1$ successive derivatives. In the case where the functions $y_1(x), y_2(x), \ldots, y_m(x)$ are linearly dependent, the corresponding Wronskian $W(x)$ must vanish identically.

We can also prove the converse result, namely that the vanishing of the determinant $W(x)$ implies the linear dependence of the functions $y_1(x), y_2(x), \ldots, y_m(x)$. In order to prove this statement, we assume that $W(x) = 0$. If we study the minor determinants of the last horizontal row, we see that they are also Wronskians, namely those formed from $m - 1$ of the functions $y_1(x), y_2(x), \ldots, y_m(x)$. We may assume that none of these minor determinants are identically equal to zero, since otherwise, by induction, $m - 1$ of the functions $y_1(x), y_2(x), \ldots, y_m(x)$ would already be linearly dependent.

The system (1.5) of equations for the coefficients $c_1, c_2, \ldots, c_m$ has a determinant which vanishes identically; therefore it has a non-trivial solution, although it must be expected that the coefficients $c_1, c_2, \ldots, c_m$ will be functions of x. None of the $c_1, c_2, \ldots, c_m$ can now vanish identically, since otherwise the system with fewer than m of the coefficients $c_1, c_2, \ldots, c_m$ would already have a non-trivial solution. In view of the fact that none of the c_r vanish, we can set, for example, $c_m = 1$. Next we differentiate successively all of the equations (1.5) except the last with respect to x. Taking the succeeding equation into account, we obtain the equations

$$c_1'y_1(x) + c_2'y_2(x) + \cdots + c_{m-1}'y_{m-1}(x) = 0,$$
$$c_1'y_1'(x) + c_2'y_2'(x) + \cdots + c_{m-1}'y_{m-1}'(x) = 0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$c_1'y_1^{(m-2)}(x) + c_2'y_2^{(m-2)}(x) + \cdots + c_{m-1}'y_{m-1}^{(m-2)}(x) = 0.$$

If now, by assumption, the Wronskian of the functions $y_1(x), y_2(x), \ldots, y_{m-1}(x)$ does not vanish, we find that $c_1' = c_2' = \cdots = c_{m-1}' = 0$, i.e. that all the $c_1, c_2, \ldots, c_{m-1}$ series are constant once we assume that c_m is constant. Hence we have proved that the functions $y_1(x), y_2(x), \ldots y_m(x)$ are linearly dependent, if their Wronskian vanishes.

The value of the Wronskian can be expressed in terms of the coefficient a_1, of the homogeneous linear differential equation

$$y^{(m)} + a_1y^{(m-1)} + a_2y^{(m-2)} + \cdots + a_{m-1}y' + a_my = 0 \qquad (1.7)$$

which is satisfied by the functions $y_1, y_2, \ldots, y_m$. In fact, we have

$$W = e^{\int a_1(x)\,dx}. \qquad (1.8)$$

This result is readily established if we form dW/dx and substitute in the differentiated Wronskian for $y_k^{(m)}$ the value $-a_1y_k^{(n-1)} - a_2y_k^{(n-2)} - \cdots - a_my_k$. In this way one finds

$$\frac{dW}{dx} = -a_1W.$$

The expression (1.8) for W follows from this equation, where the constant of integration, of course, does not vanish, since $W(x) \not\equiv 0$.

§ 4. We can now show that every solution y of a homogeneous linear nth order differential equation can be expressed in the form

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_my_m(x), \qquad (1.9)$$

where $c_1, c_2, \ldots, c_m$ are constant coefficients, if $y_1(x), y_2(x), \ldots, y_m(x)$ form a system of linearly independent functions. Since the Wronskian of the functions $y_1(x), y_2(x), \ldots, y_m(x)$ does not vanish identically, we prescribe for $y(x)$ the initial values $y(x_0), y'(x_0), y''(x_0), \ldots, y^{(m-1)}(x_0)$ in (1.9) and substitute in this equation which we differentiate with respect to x successively $m - 1$ times for x the value x_0. In this way, we obtain for $c_1, c_2, \ldots, c_m$ a soluble system of equations and the unique solution of the linear differential equation. Every such system $y_1(x), y_2(x), \ldots, y_m(x)$ of solutions which can fulfil the above task is called a *fundamental system* of solutions. If we have an mth order differential equation of any type, its general solution depends on x and m constants, which are independent of each other. Then, if we have a solution which results from a special choice of the constants, the knowledge of such a particular solution does not, as a rule, assist with regard to the establishment of the general solution. In the case of linear differential equations this is not true. As we have seen, we can in this case represent every solution as soon as we have n suitable particular solutions, i.e. a fundamental system.

It is even readily seen that the knowledge of one single particular solution brings us closer to the complete solution in the sense that we can reduce the mth order differential equation to a linear differential equation of order $m - 1$. In order to show this we set in the differential equation $y = \eta u$, where η is a particular solution of the differential equation (1.7). Now we form the successive derivatives

$$y' = u'\eta + u\eta',\ y'' = u''\eta + 2u'\eta' + u\eta'', \ldots,$$
$$y^{(m)} = u^{(m)}\eta + \binom{m}{1} u^{(m-1)}\eta' + \cdots + \binom{m}{m-1} u'\eta^{(m-1)} + u\eta^{(m)},$$

which we substitute into the differential equation. First of all, we note that these derivatives are linearly homogeneous in $u, u', u'', \ldots, u^{(m)}$. Therefore, the differential equation assumes the form

$$A_0u^{(m)} + A_1u^{(m-1)} + \cdots + A_{m-1}u' + A_mu = 0$$

in which, however, $A_m = 0$. This fact is readily verified directly, but also in the following manner: we have substituted for y the expression $y = u\eta$, where η was a particular solution of the linear differential equation. Every solution can be represented in this form, in particular those which have the form $y = \eta \cdot \text{const}$. Therefore the equation has as a solution the function $u = \text{const.}$, which can be the case in a linear differential equation only if the term with u is absent. If one writes