RIEMANN SURFACES AND RELATED TOPICS: PROCEEDINGS OF THE 1978 STONY BROOK CONFERENCE

EDITED BY

IRWIN KRA AND BERNARD MASKIT

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PREFACE

This volume contains papers and abstracts by participants of the Conference on Riemann Surfaces and Related Topics, which was held at the State University of New York at Stony Brook, June 5-9, 1978. This was the fourth in a series of conferences on more or less the same subject (Tulane 1965, Stony Brook 1969, Maryland 1973). We invited papers from all the Conference participants, with acceptance for publication subject to refereeing. All the manuscripts were indeed refereed by participants, and not all were accepted.

As usual, thanks are due to the National Science Foundation for financial support, the State University of New York at Stony Brook for its hospitality, and Princeton University Press for providing a series where these Proceedings could be published (volumes 66 and 79 contain the Proceedings of the previous two Conferences). Most of all we thank the participants in the Conference who wrote these papers and who refereed them, who gave invited lectures and seminar talks, and who talked mathematics and created the atmosphere of excitement that made our publishing effort worthwhile.

We were particularly pleased by the appearance (both at the Conference and in these Proceedings) of many new (both young and old) faces. Mathematicians from many diverse fields are now interested in Riemann Surfaces and Kleinian groups. We are delighted that the old classical theory of functions of one complex variable still shows so many signs of vitality.

I. Kra
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A GEOMETRIC PROPERTY OF BERS' EMBEDDING OF THE TEICHMÜLLER SPACE

William Abikoff

In this short note we prove a geometric property of the Bers embedding of the Teichmüller space. To fix the notation, let G be a finitely generated Fuchsian group of the first kind acting in the unit disc Δ. The Bers embedding of $T(G) = T(\Delta/G)$ represents $T(G)$ in the space B of bounded quadratic differentials ϕ for G in the exterior E of Δ. In the usual way we associate to each $\phi \in B$, the normalized solution

$$f_\phi(z) = \frac{1}{z} + \sum_{1}^{\infty} b_n(\phi) z^{-n}$$

of the Schwarzian differential equation $\{f_\phi, z\} = \phi$. It is important to note that f_ϕ, hence $b_n(\phi)$ is a holomorphic function on B. Set $G_\phi = f_\phi G f_\phi^{-1}$ and let $i: T(G) \to B$ be the Bers embedding. If $\phi \in \overline{T} = \overline{i(T(G))}$ then f_ϕ is schlicht and G_ϕ is a b-group. Let $\Lambda(G_\phi)$ denote the limit set of G_ϕ and $m(\phi)$ be the area of $\Lambda(G_\phi)$.

We prove the following

THEOREM. *If* $\phi \in \partial i(T(G))$ *and* $m(\phi) = 0$, *then* $\phi \in \partial \operatorname{Ext} i(T(G))$.

Proof. The tripartite classification of b-groups shows that if $\phi \in \partial i(T(G))$ then G_ϕ is either totally degenerate or has accidental parabolic transformations. The two cases must be handled separately.

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If G_ϕ has accidental parabolic transformations, then there is some $\gamma_\phi \in G_\phi$ so that $\tau(\phi) = \mathrm{tr}^2\,\gamma_\phi = 4$, but $\tau(\phi)$ is a nonconstant holomorphic function on B. Thus near ϕ, τ takes on all values sufficiently close to 4. It follows that there are groups G_ψ arbitrarily close to G_ϕ which have elliptic elements of infinite order. Such groups are not Kleinian and $\psi \notin \partial\, i(T(G))$.

We proceed to the case where G_ϕ is totally degenerate. Set

$$A : B \to [-\infty, \pi]$$

$$\psi \mapsto \pi\left(1 - \sum_1^\infty n|b_n(\psi)|^2\right).$$

Since $b_n(\psi)$ is holomorphic on B, A is plurisuperharmonic. Further, Gronwall's Area Theorem says that if f_ψ is schlicht then $A(\psi)$ is the area of $C \setminus f_\psi(E)$. Assume $G_\phi \in \mathrm{Int}\,\overline{i(T(G))}$. Then any holomorphic map

$$h : \overline{\Delta} \to B$$
$$0 \mapsto \phi$$

satisfies

$$A(\phi) \geq (2\pi)^{-1} \int_{\partial\Delta} A(h(e^{i\theta}))\,d\theta \geq 0\,.$$

But we may choose a holomorphic disc $h(\Delta)$ with center ϕ and intersecting $i(T(G))$ along a nontrivial boundary arc and such that $h(\overline{\Delta}) \subset \overline{i(T(G))}$. It follows from the above inequality that $A(\phi) > 0$. But for totally degenerate groups, $A(\phi) = m(\phi)$ and we have assumed $m(\phi) = 0$. We have the desired contradiction.

Some Remarks

1) The theorem is a finite dimensional version of Gehring's theorem that the universal Teichmüller space is the interior of the Schwarzians of schlicht functions.

2) At this conference, Thurston announced a proof that $m(\phi) = 0$, for all boundary groups of the Teichmüller space. This result eliminates the need for our main hypothesis.

3) The second part of the proof may be repeated verbatim to prove the following statement. Given any holomorphic mapping of the punctured disc into Teichmüller space (or equivalently, a holomorphic family of finite Riemann surfaces over the unit disc), then the puncture (or central fiber) cannot be filled in by a totally degenerate group whose limit set has zero area.

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PLANE MODELS FOR RIEMANN SURFACES ADMITTING CERTAIN HALF-CANONICAL LINEAR SERIES, PART I*

Robert D. M. Accola**

1. *Introduction*

Let W_p be a Riemann surface of genus p. In considering vanishing properties of the theta-function at half periods of the Jacobian associated with W_p, one is led naturally, via Riemann's vanishing theorem, to half-canonical linear series on W_p, that is, to linear series whose doubles are canonical. A theorem of Castelnuovo assures us that a half-canonical g^r_{p-1} must be composite if $p < 3r$, and this leads directly to the existence of automorphism of period two on W_p [2, Part III]. In this paper we are concerned with surfaces where $p = 3r$ and W_p admits a simple g^r_{p-1} (which must necessarily be half-canonical). We show that such surfaces exist for all r and we investigate the consequences. By another theorem of Castelnuovo it follows that, except for $r = 5$, the existence of a simple g^r_{p-1} on W_{3r} insures the existence of a g^1_4 without fixed points. This in turn implies that such a Riemann surface has a plane model where the half-canonical g^r_{3r-1} is easily seen. From these models one easily calculates the dimension of such Riemann surfaces in Teichmüller space. The methods developed here also allow us to characterize, for such surfaces, when the divisors of the g^1_4 are the orbits of an automorphism group which is non-cyclic of order four.

*The author wishes to express his thanks to Dr. Joseph Harris for valuable discussions concerning the material of this paper.

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It turns out that these methods also apply to W_{3r+2}'s admitting two simple half-canonical g^r_{3r+1}'s and to W_{3r+3}'s admitting four simple half-canonical g^r_{3r+2}'s whose sum is bicanonical. We shall consider these cases in Part II of this paper.

2. *Notation, definitions, and preliminary results*

A compact Riemann surface of genus p will be denoted W_p. A linear series on W_p of dimension r and degree n will be denoted g^r_n. Such a series may have fixed points, may be simple or composite, and may be complete or incomplete. For $x \in W_p$, $g^r_n - x$ will denote the linear series of degree $n-1$ of divisors of g^r_n passing through x, not counting x. If x is not a fixed point of g^r_n, then $g^r_n - x = g^{r-1}_{n-1}$.

For the convenience of the reader we include the following definitions [6, p. 257]. A linear series g^{r*}_n will be defined to be *simple* if for a general choice of x, $g^r_n - x$ is without fixed points. In this situation it is known that for a general choice of x, $g^r_n - x$ will also be simple. A linear series g^{r*}_n will be defined to be *composite* if for any choice of x, $g^r_n - x$ has fixed points. In this latter situation W_p is a t-sheeted covering of a surface of genus q, W_q, and a divisor of non-fixed points of g^r_n is a union of the fibers of the map $\phi: W_p \to W_q$. In such a case W_q admits a $g^r_{(n-f)/t}$ where f is the degree of the divisor of fixed points of g^r_n, and for x not fixed for g^r_n, $g^r_n - x$ has $t-1$ additional fixed points, the other points in the fiber of ϕ containing x. If g^r_n is complete on W_p, then so is $g^r_{(n-f)/t}$ on W_q.

If g^r_n is a linear series, a second series g^s_m is said to impose t (linear) conditions on g^r_n if there is a linear series g^{r-t}_{n-m} so that

$$g^r_n = g^s_m + g^{r-t}_{n-m}.$$

This means that if D is any divisor of g^s_m of m distinct points, then there are t points of $D, x_1, x_2, \cdots, x_t$ so that

$$g^r_n - (x_1 + x_2 + \cdots + x_t) = g^{r-t}_{n-m} + D - (x_1 + \cdots + x_t)$$

---

*Without fixed points.

has $D-(x_1+\cdots+x_t)$ among its fixed points. Also $x_1, x_2, \cdots, x_t$ impose independent conditions; that is, for each k there is a divisor in g^r_n containing all the x_i, $i=1,2,\cdots,k-1, k+1,\cdots,t$, but not containing x_k.

If g^1_m imposes one condition on g^r_n, then $g^r_n = rg^1_m + D_{n-rm}$ where D_{n-rm} is the divisor of fixed points of the composite g^r_n; for whenever a divisor of g^r_n contains a point x, it must contain all of the unique divisor of g^1_m containing x.

We will use the classical fact that since a g^1_n $(n \le p)$ without fixed points imposes $n-1$ conditions on the canonical series, it imposes at most $n-1$ conditions on any special linear series. The extension of this is that a simple special g^s_m without fixed points will impose at most $m-s$ conditions on any other special linear series whose dimension is at least $m-s$.

If g^r_n is simple $(r \ge 2)$ and without fixed points, then W_p can be realized as a curve in P^r and the hyperplane sections cut out the divisors of g^r_n. In such cases we will say that g^r_n has a k-fold singularity if the curve in P^r does. In case g^2_n is simple and without fixed points, W_p admits a plane model of degree n. If d represents the number of double points suitably counted, then

$$p = \frac{(n-1)(n-2)}{2} - d .$$

To compute the dimension R of all plane curves of degree n with s given ordinary singularities of multiplicities $k_1, k_2, \cdots, k_s$, we use the formula

$$R \ge \frac{n(n+3)}{2} - \sum_{j=1}^{s} \frac{k_j(k_j+1)}{2} .$$

Often, this formula is precise. A singularity of multiplicity k will be called a *k-fold point* of the curve or linear series.

A surface will be called *q-hyperelliptic* $(q \ge 0)$ if it is a two-sheeted cover of a surface of genus $q' \le q$. Thus rational and elliptic surfaces are q-hyperelliptic for all q.