A COURSE IN REAL ANALYSIS

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NEIL A. WEISS

实分析教程

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A Course in Real Analysis

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Preface

This is a book about real analysis, but it is not an ordinary real analysis book. Written with the student in mind, this text incorporates pedagogical techniques not often found in books at this level. The book is intended for a one-year course in real analysis at the graduate level or the advanced undergraduate level.

We bring over 50 years of combined teaching, research, and writing experience to this project. The text material has been class tested several times and has been used for independent study courses as well.

What Makes This Book Unique

This book contains many features that are unique for a real analysis text. Here are a few.

Motivation of key concepts. All key concepts are motivated. The importance of and rationale behind ideas such as measurable functions, measurable sets, and Lebesgue integration are made transparent.

Detailed theoretical discussion. Detailed proofs of most results (i.e., lemmas, theorems, corollaries, and propositions) are provided. However, to fully engage the reader, proofs or parts of proofs are often relegated to the exercises.

Illustrative examples. Following most definitions and results, one or more examples are presented that illustrate the concept or result in order to solidify it in the reader's mind and provide a concrete frame of reference. This book contains approximately 200 examples, most of which consist of several parts.

Abundant and varied exercises. The text contains over 1200 exercises, not including parts, far more than other real analysis books. Furthermore, the exercises vary widely with regard to application and level.

Applications. A diverse collection of applications appears throughout the text, some as examples and others as entire sections or chapters. For instance, applications to probability theory are ubiquitous. Other applications include those to Fourier analysis, wavelets, and measurable dynamical systems.

Careful referencing. As an aid to effective use of the book, we have consistently provided references (including page numbers) to definitions, examples, exercises, and results. Additionally, we have marked post-referenced exercises with a star (★); we strongly recommend that all such exercises be done by the reader.

Biographies. Each chapter begins with a brief biography of a famous mathematician. Besides being of general interest, these biographies help the reader obtain a perspective on how real analysis and its applications have developed.

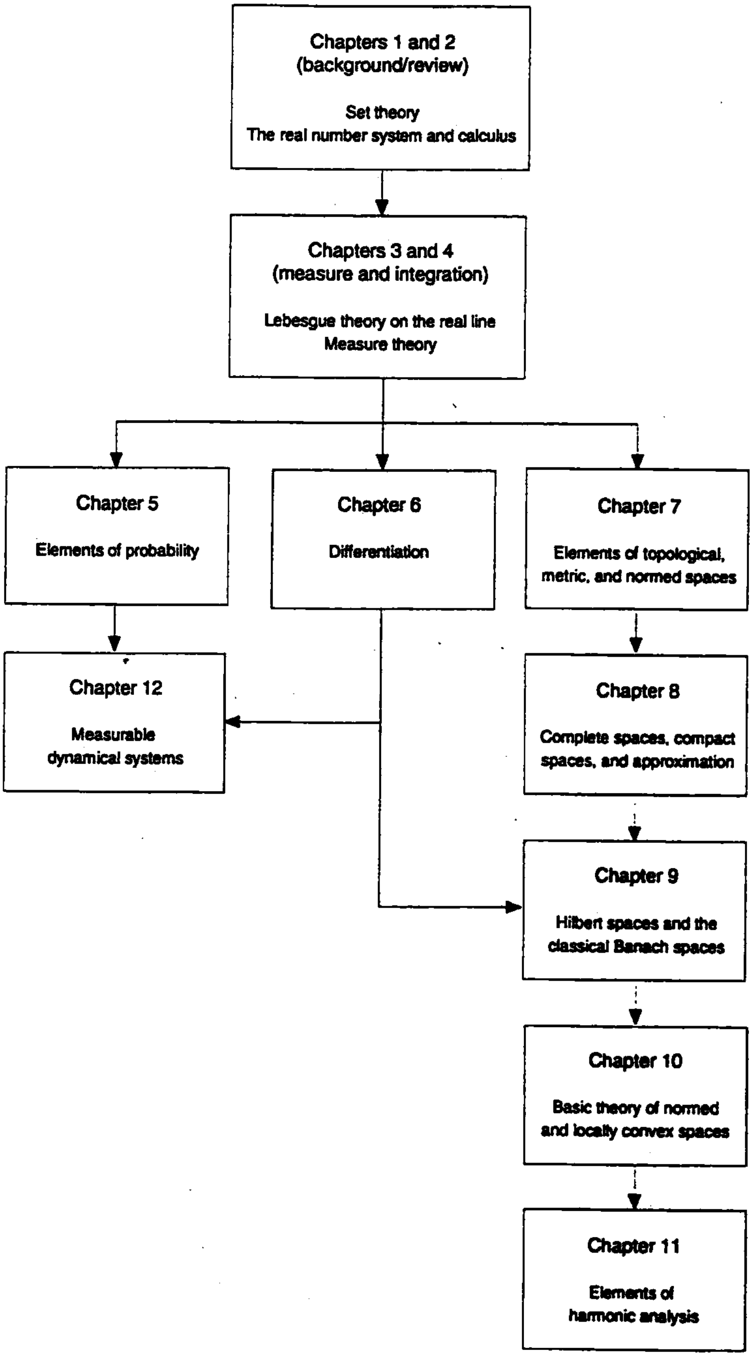
Organization

The text offers considerable flexibility in the choice of material to cover.

- Chapters 1 and 2 present prerequisite material that may be review for many but provides a common ground for all readers. At the option of the instructor, these two chapters can be covered either briefly or in detail; they can also be assigned to the students for independent reading.
- Chapters 3 and 4 present the elements of measure and integration by first discussing the Lebesgue theory on the line (Chapter 3) and then the abstract theory (Chapter 4). This material is prerequisite to all subsequent chapters.

- Chapter 5 provides an introduction to the fundamentals of probability theory, including the mathematical model for probability, random variables, expectation, and laws of large numbers. Although optional, this chapter is recommended as it provides a myriad of examples and applications for other topics.
- In Chapter 6 differentiation is discussed, both of functions and of measures. Topics examined include differentiability, bounded variation, and absolute continuity of functions, and a thorough discussion of signed and complex measures, the Radon-Nikodym theorem, decomposition of measures, and measurable transformations.
- Chapter 7 provides the fundamentals of topological and metric spaces. This chapter can be covered relatively quickly when the students have a background in topology from other courses. In addition to topics traditionally found in an introduction to topology, a discussion of weak topologies and function spaces is included.
- Completeness, compactness, and approximation comprise the topics for Chapter 8. Examined therein are the Baire category theorem, contractions of complete metric spaces, compactness in function and product spaces, and the Stone-Weierstrass theorem.
- Presented in Chapter 9 are Hilbert spaces and the classical Banach spaces. Among other things, bases and duality in Hilbert space, completeness and duality of $\mathcal{L}^p$-spaces, and duality in spaces of continuous functions are discussed.
- The basic theory of normed and locally convex spaces is given in Chapter 10. Topics include the Hahn-Banach theorem, linear operators on Banach spaces, fundamental properties of locally convex spaces, and the Krein-Milman theorem.
- Chapter 11 provides applications of previous chapters to harmonic analysis. We examine the elements of Fourier series and transforms and the $\mathcal{L}^2$-theory of the Fourier transform. In addition, an introduction to wavelets and the wavelet transform is presented.
- Chapter 12 examines measurable dynamical systems. This chapter requires the one on probability (Chapter 5) and discusses ergodic theorems, isomorphisms of measurable dynamical systems, and entropy.

The flowchart on the next page summarizes the preceding discussion and depicts the interdependence among chapters. In the flowchart, the prerequisites for a given chapter consist of all chapters having a path leading to that chapter.


Chapters 1 and 2
(background/review)
Set theory
The real number system and calculus
Chapters 3 and 4
(measure and integration)
Lebesgue theory on the real line
Measure theory
Chapter 5
Elements of probability
Chapter 6
Differentiation
Chapter 7
Elements of topological, metric, and normed spaces
Chapter 12
Measurable dynamical systems
Chapter 8
Complete spaces, compact spaces, and approximation
Chapter 9
Hilbert spaces and the classical Banach spaces
Chapter 10
Basic theory of normed and locally convex spaces
Chapter 11
Elements of harmonic analysis

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Tempe, Arizona

J.N.M
N.A.W.

Contents

PART FOUR □ Harmonic Analysis and Dynamical Systems

PART ONE

□

Set Theory, Real Numbers, and Calculus

Georg Cantor
(1845–1918)

Georg Cantor was born on March 3, 1845, in St. Petersburg, Russia. He received his doctorate in mathematics from the University of Berlin in 1867, having studied under Weierstrass, Kummer, and Kronecker. In 1869, he accepted a teaching position at the University of Halle and became a full professor in 1879.

Cantor wanted to obtain a professorship at the University of Berlin, where both pay and prestige were higher, but Kronecker, believing that much of Cantor's work (particularly his "transfinite numbers") was unsound, stood firmly in Cantor's path.

Others, however, acknowledged Cantor's genius. Cantor was an honorary member of the London Mathematical Society and received honorary doctorates from both Christiania and St. Andrews. Hilbert said Cantor's work was "... the finest product of mathematical genius and one of the supreme achievements of purely intellectual human activity."

Known as the founder of set theory, Cantor also made fundamental contributions to classical analysis. Many concepts in modern mathematics bear his name, among which are Cantor series and Cantor sets; he also developed the first usable definition of the continuum.

The controversy surrounding his work took a heavy toll on Cantor; beginning in 1884, bouts of deep depression drove him often to a sanitarium. Georg Cantor died in a psychiatric clinic at the University of Halle (where he had remained as a professor) on January 6, 1918.

1

Set Theory

In this chapter, we will introduce the fundamentals of set theory. Although some readers may be familiar with much of the material, we present this chapter as a way to provide a common ground for all readers of the text.

We will first discuss basic definitions and properties of sets. Next we will explore relationships between functions and sets, discuss Cartesian products, and introduce countability. Finally, we will examine algebras, σ-algebras, and monotone classes — special collections of sets that play a prominent role in analysis and measure theory.

1.1 BASIC DEFINITIONS AND PROPERTIES

A **set** is a collection of elements. If A is a set and x is an element (member, point) of A, then we write $x \in A$; $x \notin A$ means than x is not an element of A and, in general, we use "/" to signify negation. The symbol $\emptyset$ denotes the **empty set**, a set containing no elements.

Let A and B be sets. If every element of A is an element of B, then A is said to be a **subset** of B, denoted $A \subset B$ or $B \supset A$. Two sets, A and B, are **equal** if they contain the same elements — in other words, if