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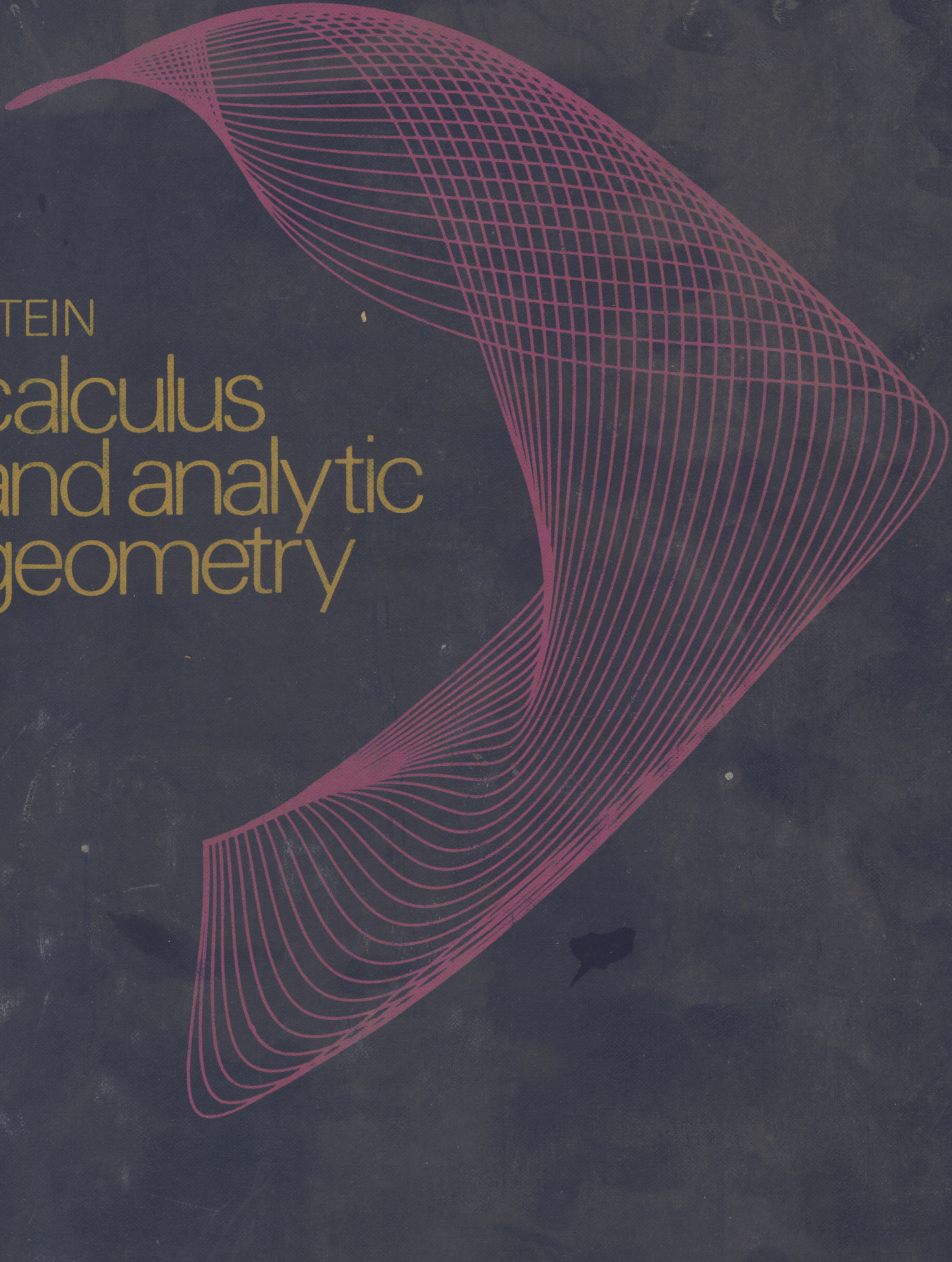
calculus and analytic geometry



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1

The two main problems of calculus

If an object traveling at constant speed moves 6 feet in 2 seconds, its speed is easy to find:

$$\text{Speed} = \frac{\text{distance}}{\text{time}} = \frac{6}{2} = 3 \text{ feet per second.}$$

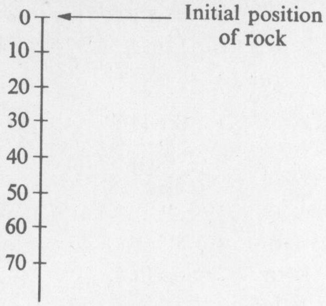
However, suppose that the object travels at a varying speed. If we know how far it travels during any period of time, how can its speed at any given instant be found? For example, assume that a rock drops $16t^2$ feet in the first t seconds of its fall, as Galileo discovered. What is its speed t seconds after its release? This question, answered in Sec. 1.1, introduces the first of the two basic concepts of the calculus, the derivative.

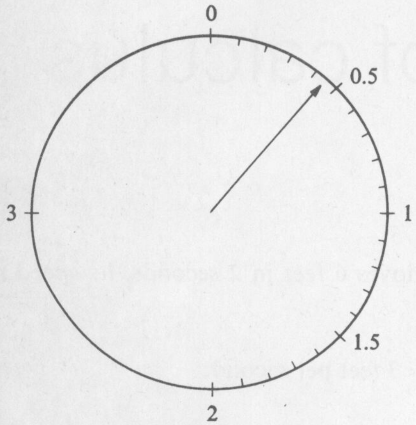
The question can be turned around: If an object travels at a varying speed and we know this speed at any instant, how can the distance it travels be found? For example, if the speed of a rocket is t^2 feet per second after t seconds, how far does it travel in the first 3 seconds? This question is answered in Sec. 1.2. It introduces the second basic concept of calculus, the definite integral.

1.1 HOW TO FIND THE VARYING SPEED FROM THE DISTANCE

Suppose that a rock, starting from rest, falls $16t^2$ feet in t seconds. What is its speed after t seconds? For the sake of simplicity, consider a specific value of t, say $t = 2$. Let us find the speed after 2 seconds.

To begin with, introduce a vertical line to record the position of the rock, as in the accompanying figure. The watch shown in the figure at the left will be used to record time. In the first half second, the rock falls $16(\frac{1}{2})^2 = 4$ feet. In the first second, the rock falls $16(1)^2 = 16$ feet. Thus, during the second half second





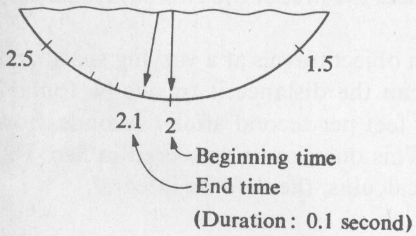
the rock falls $16 - 4 = 12$ feet, which is three times as far as it falls during the first half second.

As the rock falls, its speed increases.

To find the rock's speed after it has been falling for 2 seconds, consider the distance it travels in a brief interval after the first 2 seconds of fall. If we observe only a single instant of time (precisely 2 seconds after the rock is dropped), we cannot hope to find the speed, which depends on distance traveled. In the same way, a photographer could not hope to compute the rock's speed from a single photograph. But from two photographs taken a very short time apart, he could at least estimate the speed of the rock.

Take a specific short interval of time, say, from time $t = 2$ seconds to time 2.1 seconds, a duration of 0.1 second. To find the distance traveled, use the formula $16t^2$ and subtract:

$$\begin{aligned} 16(2.1)^2 - 16(2)^2 &= 16(4.41) - 16(4) \\ &= 70.56 - 64 \\ &= 6.56 \text{ feet.} \end{aligned}$$

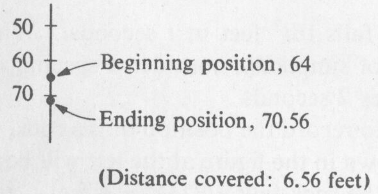


These computations show that during the interval of 0.1 second, the rock drops 6.56 feet.

A reasonable *estimate* of the speed of the rock at time $t = 2$ is therefore

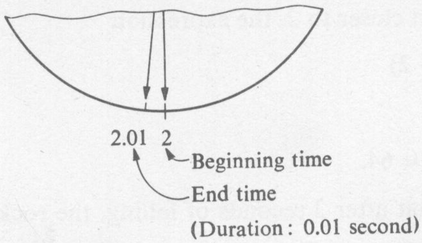
$$\frac{\text{Distance}}{\text{Time}} = \frac{6.56}{0.1} = 65.6 \text{ feet per second.}$$

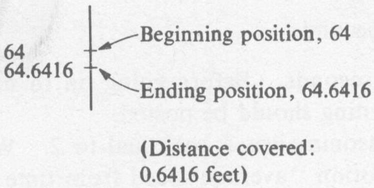
This may be thought of as the average speed during the brief interval from $t = 2$ to 2.1 seconds, a duration of 0.1 second.



Use of a shorter time interval will presumably provide a more accurate estimate. Choose, say, the time interval from $t = 2$ to 2.01 seconds, a duration of only 0.01 second. The distance the rock falls during this shorter interval of time is

$$\begin{aligned} 16(2.01)^2 - 16(2)^2 &= 16(4.0401) - 16(4) \\ &= 64.6416 - 64 \\ &= 0.6416 \text{ foot.} \end{aligned}$$





The accompanying figures show it pictorially.

Using this shorter time interval provides a more accurate estimate of the speed at time $t = 2$, namely,

$$\frac{0.6416}{0.01} = 64.16 \text{ feet per second.}$$

The average speed during this shorter interval of time is 64.16 feet per second.

Rather than compute the average speed over shorter and shorter time intervals, let us use algebra to treat the *general* short interval of time, from time = 2 to a time t_1 which is larger than 2. (Read t_1 as " t sub one.") This time interval has a duration of

$$t_1 - 2 \qquad \text{seconds.}$$

During this time the rock falls

$$16(t_1)^2 - 16(2)^2 \qquad \text{feet,}$$

and the quotient

$$\frac{16(t_1)^2 - 16(2)^2}{t_1 - 2}$$

is an estimate of the speed after 2 seconds of fall. The closer t_1 is to 2, the more closely this estimate approximates the exact speed at time 2.

A little algebra changes the quotient

$$\frac{16(t_1)^2 - 16(2)^2}{t_1 - 2}$$

into an equivalent expression much easier to work with:

$$\begin{aligned} \frac{16(t_1)^2 - 16(2)^2}{t_1 - 2} &= 16\,\frac{(t_1)^2 - (2)^2}{t_1 - 2} \\ &= 16\,\frac{(t_1 + 2)(t_1 - 2)}{t_1 - 2} \\ &= 16(t_1 + 2). \end{aligned}$$

In short,

$$\frac{16(t_1)^2 - 16(2)^2}{t_1 - 2} = 16(t_1 + 2),$$

an algebraic identity that is valid whenever t_1 is different from 2.

Note for later reference: The step in which $(t_1)^2 - (2)^2$ is replaced by

$$(t_1 + 2)(t_1 - 2)$$

follows from the algebraic identity

$$c^2 - d^2 = (c + d)(c - d).$$

It is easy to see that as t_1 gets closer and closer to 2, the expression

$$16(t_1 + 2)$$

approaches

$$16(2 + 2) = 64.$$

Therefore, it seems reasonable to claim that after 2 seconds of falling, the rock has a speed of

$$64 \text{ feet per second.}$$

We have found the speed when $t = 2$ seconds. Before going on to find a formula for the speed at any time t, a warning should be posted.

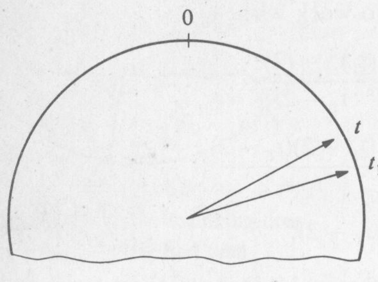
Important warning: At no step in the reasoning was t_1 set equal to 2. When $t_1 = 2$, there is no physical sense in the notion "average speed from time 2 to time 2, a duration of 0 seconds." During such a time interval the rock moves a distance of 0 feet. The quotient

$$\frac{\text{Distance}}{\text{Time}}$$

in that case becomes the meaningless expression

$$\frac{0}{0}.$$

An argument similar to that used to find the speed at $t = 2$ holds for any time t during the descent of the rock. To find the speed at time t, consider a small interval from time t to time t_1. This interval has a duration of $t_1 - t$ seconds and is illustrated in the figure at the left. During this time the rock falls



$$16(t_1)^2 - 16(t)^2 \quad \text{feet.}$$

Thus

$$\frac{16(t_1)^2 - 16(t)^2}{t_1 - t}$$

is a reasonable estimate of the speed at time t. This quotient equals

$$16\,\frac{(t_1)^2 - (t)^2}{t_1 - t}$$

$$= 16\,\frac{(t_1 + t)(t_1 - t)}{t_1 - t}$$

$$= 16(t_1 + t).$$

When t_1 is near t, the quotient is near

$$16(t + t),$$

which is

$$32t.$$

Therefore the speed of the rock after t seconds is

$$32t \qquad \text{feet per second.}$$

When $t = 2$, the general formula for the speed of the rock, $32t$, gives the value $32(2) = 64$ feet per second, in accordance with the earlier computation for that special case. The formula $32t$ shows that the speed is proportional to the time.

The preceding method, which finds the speed of the falling rock, may also be used to find the speed of an object whose motion is described by a formula other than $16t^2$. The basic approach is the same, though the specific algebraic details will be different. Exercises 10 and 11 illustrate this.

The procedure illustrated in this section is called *differentiation*. We say that differentiation of $16t^2$ yields $32t$, or that the *derivative* of $16t^2$ is $32t$.

Chapter 3 develops the ideas of this section in greater generality. It will be shown there that the speed of a moving object is just one of many applications of differentiation.

Exercises

1. This table records how far a certain object travels up to the given time:

Time, seconds	*Distance traveled up to the given time, feet*
1	4.2
1.01	4.3
1.1	5.7

(*a*) What is the average speed of the object from time $t = 1$ to 1.1?
(*b*) What is its average speed from time $t = 1$ to 1.01?

2. How far does a rock dropping $16t^2$ feet in the first t seconds fall during: (*a*) the first 0.25 second; (*b*) the first second; (*c*) the first 1.5 seconds?

3. How far does the rock of Exercise 2 fall during (*a*) the second second, (*b*) the third second, (*c*) the fourth second?

4. How far does the rock of Exercise 2 fall (*a*) from time $t = 1$ to 1.01 seconds; (*b*) from time $t = 1$ to 1.001 seconds?

5. Estimate the speed of the rock of Exercise 2 at time $t = 1$, using the data in (*a*) Exercise 4(*a*); (*b*) Exercise 4(*b*).

6. Find the speed of the rock of Exercise 2 at time $t = 1$ by considering the distance covered during the time interval from 1 second to t_1 seconds, where t_1 is larger than 1.
(*a*) How far does the rock fall during this time interval?
(*b*) How long is this time interval?
(*c*) What estimate of the speed after 1 second of fall is provided by this typical case?
(*d*) Letting t_1 approach 1, find the speed after 1 second of fall.

7. The formula developed in the text asserts that after t seconds, the speed of the rock is $32t$ feet per second. Using this formula, find its speed (in feet per second) when t is (*a*) 0; (*b*) 0.5; (*c*) 1; (*d*) 4.

8. A certain object moves t^2 feet in its first t seconds of motion. Find its speed at time t by considering short intervals of time from time t to t_1 and then letting t_1 approach t.

□

Exercise 10 concerns an object that travels t^3 feet in the first t seconds of its motion. It shows that the method of Sec. 1.1 applies quite generally. Exercise 9 presents an algebraic identity that will be needed in Exercise 10.

9. By multiplying the two factors in parentheses, show that

$$c^3 - d^3 = (c^2 + cd + d^2)(c - d).$$

10. A certain object travels t^3 feet during its first t seconds of motion.
(*a*) How far does it travel up to time t_1?
(*b*) How far does it travel from time t to time t_1?
(*c*) How long is the time interval from time t to time t_1?
(*d*) Using (*b*), (*c*), and the identity from Exercise 9, show that the average speed during the interval from time t to t_1 is

$$t_1{}^2 + t_1 t + t^2 \qquad \text{feet per second.}$$

(*e*) Use (*d*) to show that its speed at time t is $3t^2$ feet per second.
11. Use the formula in Exercise 10(*e*) to find the speed of the object in feet per second after (*a*) 0.5 second; (*b*) 1 second; (*c*) 2 seconds.
12. By multiplying the two factors in parentheses, show that

$$c^4 - d^4 = (c^3 + c^2 d + cd^2 + d^3)(c - d).$$

13. An object travels t^4 feet during its first t seconds of motion. Use the identity in Exercise 12 to show that its speed at time t is $4t^3$ feet per second.
14. An object travels t^5 feet during its first t seconds of motion. Find its speed at time t. (You will need to develop an algebraic identity for $c^5 - d^5$.)

1.2 HOW TO FIND THE DISTANCE FROM THE VARYING SPEED

If an object travels at a constant speed of 1 foot per second, then in t seconds it travels t feet. It is simple to find the distance covered by an object moving at a constant speed: just use the formula

$$\text{Distance} = \text{speed} \cdot \text{time}.$$

But what if a rocket moves in such a way that after t seconds it is traveling at t^2 feet per second? How far does it travel in the first t seconds? This question is typical of the second main problem in calculus. Observe that this question is the opposite of that raised in Sec. 1.1. Here the varying speed of an object is given, and the distance it travels is sought. In Sec. 1.1 the distances were given, and the varying speeds were sought.

First, try to get a feel for the problem by estimating how far the rocket moves during its first 3 seconds of motion, that is, up to the time $t = 3$.

The rocket moves most slowly at the beginning of its flight, since its speed at time $t = 0$ is $0^2 = 0$ feet per second. It moves most quickly at the end, when $t = 3$, and its speed is $3^2 = 9$ feet per second. Since the entire time is 3 seconds, it follows that it moves at least

$$0 \cdot 3 = 0 \text{ feet}$$

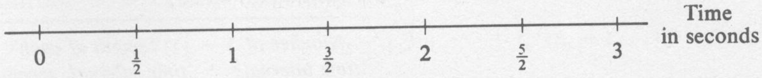
and at most

$$9 \cdot 3 = 27 \text{ feet.}$$

This tells something, but not much, about the exact distance the rocket moves in the first 3 seconds.

Though the speed of the rocket continually increases, the speed changes very little during short intervals of time. So let us divide the time interval of 3 seconds into smaller intervals of time, say, into six intervals, each of which has a duration of half a second. Estimates of the distance covered during each of these short intervals, when added together, provide an estimate of the distance covered during the first 3 seconds.

This line segment represents the time during the first 3 seconds:



(This is now more convenient to work with than the watch of Sec. 1.1.)

How far does the rocket move during the first half-second, from time $t = 0$ to time $t = \frac{1}{2}$? Since its speed keeps changing, a precise answer cannot be given immediately. However, since we know that the speed is increasing, the greatest speed of the rocket during this time interval is $(\frac{1}{2})^2$ feet per second (since its speed is t^2 feet per second at any time t). The least speed is $0^2 = 0$ feet per second, at the beginning of this initial time interval. Therefore during the first half second it travels at least

$$0^2 \cdot \tfrac{1}{2} = 0 \text{ feet}$$

and at most

$$(\tfrac{1}{2})^2 \cdot \tfrac{1}{2} = \tfrac{1}{8} \text{ foot.}$$

Similarly, during the second half second, from time $t = \frac{1}{2}$ to 1, the slowest it travels is $(\frac{1}{2})^2$ feet per second, and the fastest is $(1)^2$ feet per second. Therefore during the second half second it travels at least

$$(\tfrac{1}{2})^2 \cdot \tfrac{1}{2} = \tfrac{1}{8} \text{ foot}$$

and at most

$$(1)^2 \cdot \tfrac{1}{2} = (\tfrac{2}{2})^2 \cdot \tfrac{1}{2} = \tfrac{4}{8} \text{ foot.}$$

The remaining four half-second intervals of time can be treated similarly. It follows that during the first 3 seconds the rocket travels at least

$$0 + \tfrac{1}{8} + \tfrac{4}{8} + \tfrac{9}{8} + \tfrac{16}{8} + \tfrac{25}{8} = \tfrac{55}{8} = 6.875 \text{ feet}$$

and at most

$$\tfrac{1}{8} + \tfrac{4}{8} + \tfrac{9}{8} + \tfrac{16}{8} + \tfrac{25}{8} + \tfrac{36}{8} = \tfrac{91}{8} = 11.375 \text{ feet.}$$

	Distance covered, feet	
Time interval	*At least*	*At most*
First half second	0	1/8
Second half second	1/8	4/8
Third half second	4/8	9/8
Fourth half second	9/8	16/8
Fifth half second	16/8	25/8
Sixth half second	25/8	36/8

The table at left summarizes these computations. Thus, the distance covered during the first 3 seconds is between 6.875 and 11.375 feet. This certainly gives more information than did the first crude estimate, which was between 0 and 27 feet.

More accurate estimates can be obtained by dividing the 3-second time interval into much shorter ones: since the speed changes very little in very short time intervals, the estimates suggested by constant speed in each subinterval, namely

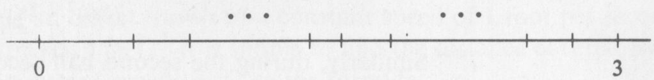
$$\text{Distance} = \text{speed} \cdot \text{time}$$

becomes more accurate. For instance, when the 3-second interval is cut into 30 intervals, each of 0.1-second duration, a calculation similar to the preceding one shows that the object travels at least 8.565 feet and at most 9.455 feet. The estimates discussed so far are recorded in this table:

Number of time intervals	*Length of each time interval, seconds*	*Lower estimate, feet*	*Upper estimate, feet*
1	3	0.0	27.0
6	0.5	6.875	11.375
30	0.1	8.565	9.455

If the time interval of 3 seconds is divided into even smaller intervals, it seems reasonable to expect that the lower and upper estimates obtained will be even closer to the actual distance traveled.

As in Sec. 1.1, the algebra of the general case is simpler than the arithmetic of the specific case. Imagine dividing the 3 seconds of time not into six, not into 30, but into n small intervals of time, where n is any positive integer. The small intervals may or may not be of equal duration.



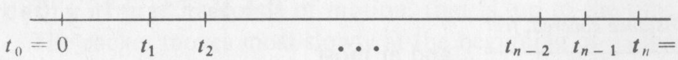
Consider a typical upper estimate. It is based on the right-hand end times of each little time interval. Call these times

$$t_1, t_2, \ldots, t_n.$$

(Read "t sub one," "t sub two," ..., "t sub n.") Observe that

$$t_n = 3.$$

For convenience, name the left-hand end time, which is 0, t_0.



During the first time interval, from time t_0 to time t_1, the rocket never goes faster than

$$t_1^2 \text{ feet per second.}$$

Since the duration of the first interval is

$$t_1 - t_0 \quad \text{seconds,}$$

the object travels at most

$$t_1{}^2(t_1 - t_0) \qquad \text{feet}$$

during that first interval.

Similarly, it travels at most

$$t_2{}^2(t_2 - t_1) \qquad \text{feet}$$

during the second time interval.

There are n intervals of time to consider. Adding the n estimates shows that during the 3 seconds the object travels at most

$$t_1{}^2(t_1 - t_0) + t_2{}^2(t_2 - t_1) + \cdots + t_n{}^2(t_n - t_{n-1}) \qquad \text{feet.}$$

We shall now show with a little algebra that this typical upper estimate is always larger than 9 feet. To begin, we need the fact that for any numbers c and d,

$$c^3 - d^3 = (c^2 + cd + d^2)(c - d),$$

an algebraic identity that can be proved by multiplying the two factors in parentheses on the right side. Assume now that c and d are nonnegative numbers and that c is larger than d; in symbols,

$$c > d \geq 0.$$

Then

$$c^2 + cd + d^2 < c^2 + c \cdot c + c^2,$$

since d on the left is replaced throughout by the larger number c on the right.

Multiplying both sides of this inequality by the positive number $c - d$ shows that

$$c^3 - d^3 < (c^2 + c^2 + c^2)(c - d),$$

or

$$c^3 - d^3 < 3c^2(c - d).$$

Therefore upon division by 3,

$$\frac{c^3 - d^3}{3} < c^2(c - d).$$

Consequently

$$c^2(c - d) > \frac{c^3 - d^3}{3}$$

or

$$c^2(c - d) > \frac{c^3}{3} - \frac{d^3}{3},$$

which is the inequality needed in the rest of the argument.

This inequality, applied to each of the n time intervals, shows that

$$t_1{}^2(t_1 - t_0) > \frac{t_1{}^3}{3} - \frac{t_0{}^3}{3}$$

$$t_2{}^2(t_2 - t_1) > \frac{t_2{}^3}{3} - \frac{t_1{}^3}{3}$$