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Random Matrices and the Six-Vertex Model

Introduction

The theory of random matrices has proven to have a wide reach into many areas of mathematics, physics, and statistics, and there are many excellent books on the topic. The book of Mehta [61] has become a classic for anyone interested in the subject, and several excellent books on random matrices have appeared in more recent years: [4] by Bai and Silverstein; [3] by Anderson, Guionnet, and Zeitouni; [42] by Forrester; [66] by Pastur and Shcherbina; the CRM volume of lectures [43] edited by Harnad; [73] by Tao; and the Oxford handbook on random matrix theory [1] edited by Akemann, Baik, and Di Francesco. See also the reviews [34] by Di Francesco, Ginsparg, and Zinn-Justin; the ones in the MSRI volume [12], edited by Bleher and Its; [33] by Di Francesco; and the forthcoming book [38] of Eynard. These books and reviews vary in scope and perspective, and they present different approaches to random matrices and their applications to combinatorics, statistics, and physics. In this book we outline a connection from random matrices to the six-vertex model of statistical physics. In particular, this model is related to the unitary matrix ensembles, which are among the most widely studied of the matrix ensembles. For unitary ensembles there is a direct connection to orthogonal polynomials on the real line, and the asymptotics of partition functions as well as local spectral statistics can be studied using the Riemann–Hilbert approach. The focus of this book is a description of the Riemann–Hilbert method for both continuous and discrete orthogonal polynomials, and applications of this approach to matrix models as well as to the six-vertex model.

The Riemann–Hilbert approach to ensembles of random matrices was initiated in the late 1990s in the papers [10] by Bleher and Its, and [29, 30] by Deift, Kriecherbauer, McLaughlin, Venakides, and Zhou, and it became a powerful tool in the theory of universality and critical phenomena in random matrices. In particular, the Riemann–Hilbert method allows for an asymptotic analysis of a wide class of orthogonal polynomials, which was a vital ingredient in the proof of universality of scaling limits for correlations of eigenvalues. The main ideas of the Riemann–Hilbert approach to orthogonal polynomials and random matrices are nicely described in the the lectures [27] by Deift. Chapter 2 of this book is adapted from the paper [29].

The six-vertex model dates back to Slater [69] in the early 1940s, and is one of the integrable models of 2-d statistical physics, see [7, 67]. The domain wall boundary conditions considered in this book were introduced by Korepin [50] in 1982. In that paper certain recursions for the partition function were derived. Subsequently these recursions were used by Izergin [46] to give an explicit determinantal formula for the partition function. This formula is the basis for the asymptotic analysis described in this book, and is known as the Izergin–Korepin formula. The relation of the Izergin–Korepin formula to ensembles of random matrices and orthogonal polynomials was discovered and used by Zinn-Justin [78, 79]. For certain values of the parameters, the relevant orthogonal polynomials are classical. In these cases, the Izergin – Korepin formula was used by Colomo and Pronko [23–26] to give expressions for the 1-, 2-, and 3-enumeration of alternating sign matrices. Outside of these special cases the orthogonal polynomials are not classical, and the Riemann – Hilbert approach was employed in a series of papers by Bleher and coauthors [8, 9, 13–15].

The general outline for the book is as follows:

- In Chapter 1 we introduce the unitary matrix ensembles and describe their connections to orthogonal polynomials and integrable systems.
- In Chapter 2 we discuss the Riemann – Hilbert (RH) approach to random matrix ensembles, adapted from the original approach of the paper [29] and the book [27]. We give general formulas for asymptotics of recurrence coefficients for orthogonal polynomials, and give a proof of the universality of the sine and Airy kernels in the bulk and at the edge, respectively, of the spectrum.
- In Chapter 3 we consider an extension of the RH approach to discrete orthogonal polynomials on an infinite lattice, which was originally developed in the book [5] of Baik, Kriecherbauer, McLaughlin, and Miller for discrete orthogonal polynomials on a finite lattice, and then extended to an infinite lattice in the paper [16] by Bleher and Liechty. Again we give general formulas for asymptotics of recurrence coefficients. Universality of the local correlations in the discrete orthogonal polynomial ensemble is discussed, and we give a proof of the scaling limit of the correlation kernel at the point which separates a band from a saturated region.
- In Chapter 4 we introduce the six vertex model with with domain wall boundary conditions.
- In Chapter 5 we derive the Izergin – Korepin formula for the partition function of the six vertex model with with domain wall boundary conditions. The proof is based on the Yang – Baxter equations, and we follow the elegant approach of the papers [51, 55].
- In Chapters 6 – 8 we obtain the large n asymptotic formulas for the partition function in different phase regions on the phase diagram. These chapters follow the works [9, 13, 15]. The methods of Chapters 2 and 3 are applied, and all details of the analysis are presented.
- In Chapter 9 we discuss the asymptotics of the partition function on the critical lines between the phases, as well as the phase transitions. The results of the papers [8, 14] for the partition function on the critical lines are discussed, but we do not present detailed proofs in this book.

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Contents

CHAPTER 1

Unitary Matrix Ensembles

1.1. Unitary ensemble with real analytic interaction

Recall that a matrix M is Hermitian if $M = M^*$ (where $M^* = \overline{M^T}$), so that $M_{kj} = \overline{M_{jk}}$. Of course, any Hermitian matrix must have real entries along the diagonal, whereas the entries below the diagonal are completely determined by the entries above the diagonal. It follows that, in order to count the real dimension of the space of $N \times N$ Hermitian matrices, we should count the number of entries along the diagonal, N, and twice the number of entries above the diagonal to account for real and imaginary parts. Thus, if $\mathcal{H}_N$ is the space of $N \times N$ Hermitian matrices, then its real dimension is equal to

$$\dim \mathcal{H}_N = N + 2[1 + 2 + \cdots + N - 1] = N + N(N-1) = N^2. \tag{1.1.1}$$

The space $\mathcal{H}_N$ is a real Hilbert space with respect to the scalar product

$$\begin{aligned}(L, M) &= \operatorname{Re}\operatorname{Tr}(LM^*) = \sum_{j,k=1}^{N} \operatorname{Re}\big(L_{jk}\overline{M_{jk}}\big) \\ &= \sum_{j=1}^{N} L_{jj}M_{jj} + 2\sum_{j>k}^{N}[(\operatorname{Re} L_{jk})(\operatorname{Re} M_{jk}) + (\operatorname{Im} L_{jk})(\operatorname{Im} M_{jk})].\end{aligned} \tag{1.1.2}$$

Being a subspace of the space of $N \times N$ matrices with complex entries, $\mathcal{H}_N$ embeds naturally into $\mathbb{C}^{N^2}$. The Euclidean distance inherited from this embedding is given as

$$\begin{aligned}\operatorname{dist}(L, M) = \|L - M\| &= \Big(\sum_{j,k=1}^{N} |L_{jk} - M_{jk}|^2\Big)^{1/2} \\ &= \Big(\sum_{j=1}^{N}|L_{jj} - M_{jj}|^2 + 2\sum_{j>k}^{N}|L_{jk} - M_{jk}|^2\Big)^{1/2}.\end{aligned} \tag{1.1.3}$$

The scalar product (L, M) and the distance $\operatorname{dist}(L, M)$ are invariant with respect to the conjugation by any unitary matrix $U \in \mathrm{U}(N)$,

$$M \to U^{-1}MU, \quad U \in \mathrm{U}(N). \tag{1.1.4}$$

Let $\mathrm{d}M$ be the N^2-dimensional Lebesgue measure,

$$\mathrm{d}M = \prod_{j=1}^{N} \mathrm{d}M_{jj} \prod_{j\neq k}^{N} \mathrm{d}\operatorname{Re} M_{jk}\, \mathrm{d}\operatorname{Im} M_{jk}. \tag{1.1.5}$$

We will consider the probability distribution on $\mathcal{H}_N$ given by

$$\mathrm{d}\mu_N(M) = \frac{1}{Z_N}\, \mathrm{e}^{-N \operatorname{Tr} V(M)}\, \mathrm{d}M, \tag{1.1.6}$$

where $V(x)$ is a real analytic function satisfying the growth condition that

$$\frac{V(x)}{\log(|x|^2+1)} \to +\infty \quad \text{as } |x| \to \infty. \tag{1.1.7}$$

This growth condition is a technical condition for subsequent analysis. Indeed, the reader may simply think of V as a polynomial of even degree and with positive leading coefficient, in which case it is clear what is meant by the matrix $V(M)$ in (1.1.6). If V is not a polynomial, then the matrix $V(M)$ can be understood by applying V to the spectrum of M. That is, M can be diagonalized as $M = U\Lambda U^*$, where U is a unitary matrix and $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$ is the matrix of eigenvalues, and we can then define

$$V(M) = U \operatorname{diag}\big(V(\lambda_1), V(\lambda_2), \dots, V(\lambda_N)\big) U^*. \tag{1.1.8}$$

Notice then that $\operatorname{Tr} V(M)$ is invariant with respect to unitary conjugations given in (1.1.4). Since the distance $\operatorname{dist}(L, M)$ induces the measure $2^{N(N-1)/2}\, \mathrm{d}M$ on $\mathcal{H}_N$, the Lebesgue measure $\mathrm{d}M$ is invariant with respect to unitary conjugations (1.1.4) as well. It follows that the distribution $\mathrm{d}\mu_N(M)$ is invariant with respect to any unitary conjugation (1.1.4), hence the name of the ensemble. The normalizing constant Z_N, called the partition function, is defined such that μ_N is a probability measure. That is, it is the matrix integral

$$Z_N = \int_{\mathcal{H}_N} \mathrm{e}^{-N \operatorname{Tr} V(M)}\, \mathrm{d}M. \tag{1.1.9}$$

Example (Gaussian unitary ensemble). For $V(M) = M^2$, the measure μ_N is the probability distribution of the Gaussian unitary ensemble (GUE). This is the oldest and most well known of the invariant matrix ensembles. In this case,

$$\operatorname{Tr} V(M) = \operatorname{Tr} M^2 = \sum_{j,k=1}^{N} M_{kj} M_{jk} = \sum_{j=1}^{N} M_{jj}^2 + 2\sum_{j>k} |M_{jk}|^2, \tag{1.1.10}$$

hence

$$\mathrm{d}\mu_N^{\mathrm{GUE}}(M) = \frac{1}{Z_N^{\mathrm{GUE}}} \prod_{j=1}^{N} \big(\mathrm{e}^{-N M_{jj}^2}\big) \prod_{j>k} \big(\mathrm{e}^{-2N|M_{jk}|^2}\big)\, \mathrm{d}M, \tag{1.1.11}$$

so that the matrix elements in GUE are independent Gaussian random variables. The partition function of GUE is evaluated as

$$\begin{aligned} Z_N^{\mathrm{GUE}} &= \int_{\mathcal{H}_N} \prod_{j=1}^{N} \big(\mathrm{e}^{-N M_{jj}^2}\big) \prod_{j>k} \big(\mathrm{e}^{-2N|M_{jk}|^2}\big)\, \mathrm{d}M \\ &= \left(\frac{\pi}{N}\right)^{N/2} \left(\frac{\pi}{2N}\right)^{N(N-1)/2} = \left(\frac{\pi}{N}\right)^{N^2/2} \left(\frac{1}{2}\right)^{N(N-1)/2}. \end{aligned} \tag{1.1.12}$$

The GUE is somewhat special in that it lies at the intersection of the invariant ensembles, which are invariant with respect to some sort of matrix conjugation (in this case unitary conjugation), and the Wigner ensembles, for which the matrix entries are independent. If the function $V(x)$ is not quadratic, then the matrix entries become dependent.

1.2. Ensemble of eigenvalues

The central topic in random matrix theory is the distribution of the eigenvalues of a random matrix. We can write a formula for the distribution of eigenvalues of an Hermitian matrix M from distribution (1.1.6) by writing M in terms of its eigenvalues and eigenvectors, so that $M = U\Lambda U^*$, where U is a unitary matrix (of eigenvectors) and $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$ is the matrix of eigenvalues. In order to make this map one-to-one, let us consider $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ to be ordered, so that $\lambda_1 \le \lambda_2 \le \dots \le \lambda_N$. In fact, dropping a set of measure zero, we can assume $\lambda_j \ne \lambda_k$ for $j \ne k$, and thus consider λ in the Weyl chamber

$$\lambda_1 < \lambda_2 < \dots < \lambda_N. \tag{1.2.1}$$

Also, instead of $U \in \mathrm{U}(N)$ we may consider matrix UD, where

$$D = \operatorname{diag}(\mathrm{e}^{\mathrm{i}\vartheta_1}, \dots, \mathrm{e}^{\mathrm{i}\vartheta_N}) \in \mathrm{D}(N)$$

is any diagonal unitary matrix. Consider therefore the equivalence class

$$\widetilde{U} = \{UD, D \in \mathrm{D}(N)\}, \tag{1.2.2}$$

and the homogeneous space $\widetilde{\mathrm{U}}(N)$ of the equivalence classes $\widetilde{U}$. Then the map

$$(\widetilde{U}, \Lambda) \mapsto M = U\Lambda U^*, \quad U \in \widetilde{U}, \tag{1.2.3}$$

is one-to-one, and we may consider its Jacobian

$$J = \frac{\mathrm{d}M}{\mathrm{d}\widetilde{U}\,\mathrm{d}\lambda}, \tag{1.2.4}$$

where $\mathrm{d}\widetilde{U}$ is the projection of the Haar measure on $\mathrm{U}(N)$ onto $\widetilde{\mathrm{U}}(N)$ and $\mathrm{d}\lambda = \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_N$. Since $\mathrm{d}M$ is invariant with respect to the unitary conjugations, and $\mathrm{d}\widetilde{U}$ is invariant with respect to the unitary left shifts, the Jacobian J does not depend on $\widetilde{U}$. Its dependence on $\lambda = (\lambda_1, \dots, \lambda_N)$ is described as follows.

Proposition 1.2.1 (Weyl's formula). *For some constant $C_N > 0$,*

$$J = C_N \prod_{j<k} |\lambda_k - \lambda_j|^2. \tag{1.2.5}$$

PROOF. Since J does not depend on $\widetilde{U}$, it suffices to evaluate J at $U = I$, i.e., at $\widetilde{U} = \tilde{I} = \mathrm{D}(N)$. In a small neighborhood of $\tilde{I}$, the elements $\widetilde{U}$ are uniquely represented by unitary matrices $U = \mathrm{e}^A$, where $A^* = -A$. As $A \to 0$,

$$U = \mathrm{e}^A = I + A + \mathcal{O}(A^2), \quad U^* = \mathrm{e}^{-A} = I - A + \mathcal{O}(A^2), \tag{1.2.6}$$

and

$$M = U\Lambda U^{-1} = \Lambda + [A, \Lambda] + \mathcal{O}(A^2), \tag{1.2.7}$$

so that

$$M_{ii} = \lambda_i + \mathcal{O}(A^2), \qquad M_{ij} = (\lambda_j - \lambda_i)A_{ij} + \mathcal{O}(A^2), \quad i < j. \tag{1.2.8}$$

This implies that at $A = 0$,

$$\begin{aligned} \frac{\partial M_{ii}}{\partial \lambda_k} &= \delta_{ik}, & \frac{\partial M_{ii}}{\partial A_{kl}} &= 0, \\ \frac{\partial M_{ij}}{\partial \lambda_k} &= 0, & \frac{\partial M_{ij}}{\partial A_{kl}} &= (\lambda_j - \lambda_i)\delta_{ik}\delta_{jl}, \quad i < j, \end{aligned} \tag{1.2.9}$$

hence the Jacobian matrix $\nabla_{\lambda,A} M$ is diagonal and its determinant is equal to

$$J = \prod_{j<k} |\lambda_k - \lambda_j|^2. \tag{1.2.10}$$

Since at $A = 0$,

$$d\widetilde{U} = C_N \, dA, \quad C_N > 0, \tag{1.2.11}$$

formula (1.2.5) follows. □

It follows that, if we are interested only in eigenvalues, the eigenvectors may be integrated out, and we find that the distribution of eigenvalues of M with respect to the ensemble μ_N is given as

$$d\mu_N(\lambda) = \frac{1}{\widetilde{Z}_N} \prod_{j>k} (\lambda_j - \lambda_k)^2 \prod_{j=1}^{N} e^{-NV(\lambda_j)} \, d\lambda, \tag{1.2.12}$$

where

$$\widetilde{Z}_N = \int \prod_{j>k} (\lambda_j - \lambda_k)^2 \prod_{j=1}^{N} e^{-NV(\lambda_j)} \, d\lambda, \quad d\lambda = d\lambda_1 \cdots d\lambda_N. \tag{1.2.13}$$

Since we have considered λ in the Weyl chamber (1.2.1), this gives a measure on ordered eigenvalues, and the integral (1.2.13) is over the Weyl chamber. However, clearly (1.2.12) is symmetric in λ, and therefore the measure can be lifted to an ensemble of unordered eigenvalues, where the integral in (1.2.13) is then understood to be over $\mathbb{R}^N$.

Notice that

$$Z_N = \widetilde{Z}_N \frac{C_N \operatorname{Vol}(\widetilde{\mathrm{U}}(N))}{N!}, \tag{1.2.14}$$

and thus the ratio $\widetilde{Z}_N / Z_N$ does not depend on the potential V. We can calculate this ratio in the case of the GUE, and the result will hold for any unitary ensemble given by (1.1.6). Indeed, for GUE,

$$d\mu_N^{\mathrm{GUE}}(\lambda) = \frac{1}{\widetilde{Z}_N^{\mathrm{GUE}}} \prod_{j>k} (\lambda_j - \lambda_k)^2 \prod_{j=1}^{N} e^{-N\lambda_j^2} \, d\lambda, \tag{1.2.15}$$

where

$$\widetilde{Z}_N^{\mathrm{GUE}} = \int \prod_{j>k} (\lambda_j - \lambda_k)^2 \prod_{j=1}^{N} e^{-N\lambda_j^2} \, d\lambda. \tag{1.2.16}$$

The constant $\widetilde{Z}_N^{\mathrm{GUE}}$ is a Selberg integral, and its exact value is

$$\widetilde{Z}_N^{\mathrm{GUE}} = \frac{(2\pi)^{N/2}}{(2N)^{N^2/2}} \prod_{k=1}^{N} k!, \tag{1.2.17}$$

see, e.g., [61]. A proof of formula (1.2.17) from the discrete string equations for orthogonal polynomials is given subsequently in Section 1.3. We therefore have that the partition functions Z_N and $\widetilde{Z}_N$ are related as

$$\frac{\widetilde{Z}_N}{Z_N} = \frac{\widetilde{Z}_N^{\mathrm{GUE}}}{Z_N^{\mathrm{GUE}}} = \frac{1}{\pi^{N(N-1)/2}} \prod_{k=1}^{N} k!. \tag{1.2.18}$$

One of the main problems in random matrix theory is to evaluate the large N asymptotics of the partition function $\widetilde{Z}_N$ and of the correlations between eigenvalues.

From (1.2.12), the joint probability density function for the eigenvalues is given by

$$p_N(x_1,\dots,x_N) = \widetilde{Z}_N^{-1} \prod_{j>k}(x_j - x_k)^2 \prod_{j=1}^{N} e^{-NV(x_j)}. \tag{1.2.19}$$

Integrating out $(N-m)$ variables, we obtain the marginal probability density function for m eigenvalues,

$$p_{mN}(x_1,\dots,x_m) = \int_{\mathbb{R}^{N-m}} p_N(x_1,\dots,x_N)\,\mathrm{d}x_{m+1}\cdots\mathrm{d}x_N. \tag{1.2.20}$$

The m-point correlation function is then defined as

$$R_{mN}(x_1,\dots,x_m) := \frac{N!}{(N-m)!}p_{mN}(x_1,\dots,x_m), \tag{1.2.21}$$

see, e.g., [2,3,36,61]. Remarkably, these correlation functions can all be expressed in terms of a system of orthogonal polynomials. Let $\{P_k(x)\}_{k=0}^{\infty}$ be the system of monic orthogonal polynomials defined from the orthogonality condition

$$\int_{-\infty}^{\infty} P_j(x)P_k(x)\mathrm{e}^{-NV(x)}\,\mathrm{d}x = h_k\delta_{jk}, \tag{1.2.22}$$

for some system of normalizing constants $\{h_k\}_{k=0}^{\infty}$. Existence and uniqueness of these polynomials is guaranteed by condition (1.1.7). Define also the functions

$$\psi_k(x) = \frac{1}{h_k^{1/2}}P_k(x)\mathrm{e}^{-NV(x)/2}, \tag{1.2.23}$$

which form an orthonormal basis in $L^2(\mathbb{R}^1)$. We have the following proposition.

Proposition 1.2.2. *The correlation function* (1.2.21) *has the determinantal form*

$$R_{mN}(x_1,\dots,x_m) = \det\big(K_N(x_k,x_l)\big)_{k,l=1}^{m}, \tag{1.2.24}$$

where

$$K_N(x,y) = \sum_{n=0}^{N-1}\psi_n(x)\psi_n(y). \tag{1.2.25}$$

Furthermore, the partition function $\widetilde{Z}_N$ can be written in terms of the orthogonal polynomials (1.2.22) *as*

$$\widetilde{Z}_N = N!\prod_{j=0}^{N-1} h_j. \tag{1.2.26}$$

An ensemble whose correlations can be expressed by such a determinantal formula is called a *determinantal point process*, see [18,20,44,45]. In particular notice that the one point correlation function enables us to write the density of eigenvalues on the real line, which we notate ρ_N in the simple form

$$\rho_N(x) := \frac{R_{1N}(x)}{N} = \frac{K_N(x,x)}{N}. \tag{1.2.27}$$

Before proving Proposition 1.2.2, let us point out some unique properties of the function K_N. Observe that K_N is the kernel of the projection operator onto the N-dimensional space generated by the first N functions ψ_n, $n = 0, \dots, N-1$. The function $K_N(x,y)$ is called the *reproducing kernel* and it has the following properties:

$$\begin{aligned} \int_{\mathbb{R}} K_N(x,x)\,\mathrm{d}x &= N, \\ \int_{\mathbb{R}} K_N(x,y)K_N(y,z)\,\mathrm{d}y &= K(x,z). \end{aligned} \tag{1.2.28}$$

Indeed, by (1.2.25),

$$\int_{\mathbb{R}} K_N(x,x)\,\mathrm{d}x = \sum_{j=0}^{N-1} \int_{\mathbb{R}} \psi_j(x)^2\,\mathrm{d}x = \sum_{j=0}^{N-1} 1 = N, \tag{1.2.29}$$

and

$$\begin{aligned} \int_{\mathbb{R}} K_N(x,y)K_N(y,z)\,\mathrm{d}y &= \sum_{j,k=0}^{N-1} \int_{\mathbb{R}} \psi_j(x)\psi_j(y)\psi_k(y)\psi_k(z)\,\mathrm{d}y \\ &= \sum_{j=0}^{N-1} \psi_j(x)\psi_j(z) = K(x,z). \end{aligned} \tag{1.2.30}$$

Let us now prove formula (1.2.26) for the partition function. Recall the formula for the Vandermonde determinant,

$$\det\left[x_j^{k-1}\right]_{j,k=1}^{N} = \prod_{\substack{j,k=1 \\ j<k}}^{N} (x_k - x_j). \tag{1.2.31}$$

The main point in the proof of (1.2.26) is that the function $p_N(x_1, \dots, x_N)$ in the integrand of (1.2.16) is the product of the square of the Vandermonde determinant and factors which are independent and identical on each of the coordinates x_j. The form of the Vandermonde matrix and multilinearity of the determinant function allow us to replace the jth row of the Vandermonde matrix with any monic polynomial of degree $(j-1)$. In particular, we may use the orthogonal polynomials described in (1.2.22), so that

$$\begin{aligned} \widetilde{Z}_N &= \int_{\mathbb{R}^N} p_N(x_1, \dots, x_N)\,\mathrm{d}x_1 \cdots \mathrm{d}x_N \\ &= \int_{\mathbb{R}^N} \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_N \\ x_1^2 & x_2^2 & \dots & x_N^2 \\ \vdots & \vdots & & \vdots \\ x_1^{N-1} & x_2^{N-1} & \dots & x_N^{N-1} \end{pmatrix}^2 \prod_{j=1}^{N} \mathrm{e}^{-NV(x_j)}\,\mathrm{d}x_j \end{aligned} \tag{1.2.32}$$