FUNCTION THEORETIC METHODS IN PARTIAL DIFFERENTIAL EQUATIONS

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Preface

In traditional treatments of the partial differential equations of mathematical physics, particular stress is laid on solving boundary value problems and initial-boundary value problems. The reasons are that these problems were the *natural* ones to consider in classical physics, i.e., in fluid dynamics, elasticity, plasticity, and electromagnetics. In quantum mechanics and quantum field theory, however, one is usually not concerned with solving boundary value problems, but with investigating the *analytic* properties of solutions of partial differential equations.

The purpose of this monograph is to present a treatment of the analytic theory of partial differential equations which will be accessible to applied mathematicians, physicists, and quantum chemists. It is assumed that the reader approaching this subject already has a knowledge of functions of one complex variable, and an acquaintance with the equations of classical mathematical physics. However, it is not assumed that the reader has any knowledge of the theory of functions of several complex variables. In order to have the book self-contained, an introductory chapter to the local theory of several complex variables is included. The reader who has some acquaintance with the subject may skip this chapter and refer back to it as needed.

The point of view taken in this monograph is essentially that of the theory of integral operators. These procedures not only enable us to determine solutions of partial differential equations, but to translate most of the theorems of one and several complex variables to the theory of partial differential equations.

In the last chapter my "envelope method," which is a generalization of the idea used by Hadamard in the proof of his multiplication of singularities theorem, is applied to scattering problems in quantum mechanics and quantum field theory.

The material presented in this monograph is based on seminars and lectures given by me at Indiana University in connection with the Mathematical Physics Program, and at the Institute for Fluid Dynamics and Applied Mathematics, University of Maryland (1961–1965). I wish at this time to express my gratitude to Professor Alexander Weinstein for providing a pleasant and stimulating mathematical environment that encouraged my individual research and study at the Institute.

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R. P. GILBERT

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I

An introduction to the theory of several complex variables

1. *Fundamentals of the Local Theory*

We begin by considering functions defined in an open region $\mathfrak{D}$ which is a subset of the space of n complex variables $\mathbb{C}^n$, i.e., the set of all n-tuples $(z_1, \ldots, z_n)$ where $z_k = x_k + iy_k$ and $x_k, y_k \in (-\infty, +\infty)$. Unless otherwise stated we shall assume that the function $f(z) \equiv f(z_1, \ldots, z_n)$ is single valued, and that $\mathfrak{D}$ is connected. Our definition of continuity is the usual one, i.e., $f(z)$ is continuous at $z^0 \in \mathfrak{D}$ if given an arbitrary $\varepsilon > 0$ we have

$$|f(z_1{}^0 + \Delta z_1, \ldots, z_n{}^0 + \Delta z_n) - f(z_1{}^0, \ldots, z_n{}^0)| < \varepsilon$$

provided that the euclidean norm of Δz is sufficiently small, i.e., $\|\Delta z\|_e = (|\Delta z_1|^2 + \cdots + |\Delta z_n|^2)^{1/2} < \delta(z)$. If $\delta(z)$ is independent of z for all $z \in \mathfrak{D}$ then we say $f(z)$ is uniformly continuous in $\mathfrak{D}$.

Definition *A complex-valued function $f(z)$ defined in a domain $\mathfrak{D}$ contained in the space of n-complex variables is said to be Weierstrass holomorphic in the domain $\mathfrak{D}$ if for each point $a \in \mathfrak{D}$ the function can be expanded as a power series of the form*

$$\begin{aligned} f(z) &= \sum_{m=0}^{\infty} c_m (z-a)^m \\ &\equiv \sum_{m_1, \ldots, m_n = 0}^{\infty} c_{m_1 \cdots m_n} (z_1 - a_1)^{m_1} \cdots (z_n - a_n)^{m_n}, \end{aligned} \tag{1.1.1}$$

which converges in some nonvoid neighborhood of a. (The point $(z_1, \ldots, z_n) = (a_1, \ldots, a_n)$ is referred to as the *center* of the power series expansion.)

We now show that if the n-fold series (1.1.1) converges in some order at the point $z = z^0 \neq a$ it converges absolutely and uniformly to the same value independent of the order of summation for all z that are contained in the "polydisk"

$$\{z \,|\, |z_k - a_k| \leq |z_k{}^0 - a_k| - \varepsilon_k;\ \varepsilon_k > 0, \text{ and } k = 1, 2, \ldots, n\}. \tag{1.1.2}$$

Since (1.1.1) converges when summed in a certain order as a simple series, it is necessary that $|c_m(z^0 - a)^m| \leq B < \infty$ for all values of the indices, $m \equiv (m_1, \ldots, m_n)$. Setting $|z^0 - a| = r \equiv r_1 \cdot r_2 \cdot \cdots \cdot r_n$ one has $|c_m| \leq B/r^m$, from which it follows that

$$\sum_{m=0}^{\infty} \left| c_m(z-a)^m \right| \leq B \sum_{m=0}^{\infty} \left| \frac{z-a}{r} \right|^m = \frac{B}{\prod_{k=1}^{n} \left(1 - \frac{|z_k - a_k|}{r_k}\right)}, \tag{1.1.3}$$

and hence it is seen that (1.1.1) converges uniformly and absolutely in the set (1.1.2). (The interior of the set (1.1.2) is called an n-circular polycylindrical region.) Since the series (1.1.1) converges absolutely in the polycylinder (1.1.2) it may be summed as a simple series in any order and converges to the same value.

Definition *We shall say that a complex-valued function $f(z)$ is holomorphic in the sense of Cauchy–Riemann in the domain $\mathfrak{D} \subset \mathbb{C}^n$ if the first partial derivatives*

$$\frac{\partial f(z)}{\partial z_k} = \lim_{\Delta z_k \to 0} \frac{f(z_1, \ldots, z_k + \Delta z_k, \ldots, z_n) - f(z_1, \ldots, z_n)}{\Delta z_k} \qquad (k = 1, 2, \ldots, n), \tag{1.1.4}$$

exist at each point $z \in \mathfrak{D}$, and are continuous.

If one separates $f(z)$ into its real and imaginary parts, $u = \operatorname{Re} f(z)$, $v = \operatorname{Im} f(z)$, and if $f(z)$ is holomorphic in the sense of Cauchy–Riemann one has that

$$\frac{\partial u}{\partial x_k} = \frac{\partial v}{\partial y_k}, \quad \text{and} \quad \frac{\partial u}{\partial y_k} = -\frac{\partial v}{\partial x_k}, \tag{1.1.5}$$

with $z_k = x_k + iy_k$ $(k = 1, 2, \ldots, n)$. In other words, "Cauchy–Riemann holomorphic" is equivalent to saying that $f(z)$ is holomorphic in each variable separately while the other variables are held fixed. If we formally introduce the variables $z_j = x_j + iy_j$ and $\bar{z}_j = x_j - iy_j$ then (1.1.5) is seen to be equivalent to the system of equations, $\partial f/\partial \bar{z}_j = 0$ $(j = 1, 2, \ldots, n)$. The exact meaning of this statement will be made clear shortly; however, accepting this statement formally implies the result that *each Weierstrass holomorphic function is indeed also holomorphic in the Cauchy–Riemann sense.* This follows directly from the fact that in its polycylinder of convergence the power series (1.1.1) may be summed as a simple series, and hence if all the z_k except z_j $(k \neq j)$ are held fixed, it represents a holomorphic function in the z_j variable. We shall see in what follows that the proof of the converse is not so obvious.

Definition *An ordinary polycylindrical region (or polycylinder) in $\mathbb{C}^n$ is the Cartesian product of n bounded, simply connected, regions $\mathfrak{D}_k$ in the z_k-planes.*

Theorem 1.1.1 *Let $f(z)$ be Cauchy–Riemann holomorphic and continuous in the closure of the polycylinder, $\mathfrak{D} \equiv \prod_{k=1}^{n} \mathfrak{D}_k$. Furthermore, let the boundaries, $\partial\mathfrak{D}_k$, of $\mathfrak{D}_k$ be piecewise smooth curves. Then if z is an interior point of $\mathfrak{D}$ we have*

$$f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{\mathfrak{S}_n} \frac{f(\zeta)}{\prod_{k=1}^{n} (\zeta_k - z_k)} \, d\zeta_1 \cdots d\zeta_n, \tag{1.1.6}$$

where $\mathfrak{S}_n \equiv \prod_{k=1}^{n} \partial\mathfrak{D}_k$ is called the "skeleton" or "distinguished boundary" of $\mathfrak{D}$.

Proof We prove this theorem by making repeated application of Cauchy's formula for one variable. In the case $n = 2$ we have for $z_2 \in \mathfrak{D}_2$ and fixed

$$f(z_1, z_2) = \frac{1}{2\pi i} \int_{\partial\mathfrak{D}_1} \frac{f(\zeta_1, z_2)}{\zeta_1 - z_1} \, d\zeta_1$$

and hence we obtain the iterated integral

$$f(z_1, z_2) = \left(\frac{1}{2\pi i}\right)^2 \int_{\partial\mathfrak{D}_1} d\zeta_1 \int_{\partial\mathfrak{D}_2} d\zeta_2 \, \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)}$$

for $(z_1, z_2) \in \mathfrak{D}$. If the distance of z_k from the boundary $\partial\mathfrak{D}_k$ is greater than some $\delta_k > 0$, the integrand is absolutely integrable and we may rewrite this as the double integral

$$f(z) = \left(\frac{1}{2\pi i}\right)^2 \int_{\mathfrak{S}_2} \frac{f(\zeta)\, d\zeta_1\, d\zeta_2}{(\zeta_1 - z_1)(\zeta_2 - z_2)}.$$

The proof for n variables follows by induction.

Theorem 1.1.2 *Let $\{f_n(z)\}_{n=1}^{\infty}$ be a sequence of functions, Cauchy–Riemann holomorphic in $\mathfrak{D} \subset \mathbb{C}^n$. Furthermore, let the partial sums $F_n(z) = f_1(z) + \cdots + f_n(z)$ converge uniformly in $\mathfrak{D}$ to $F_0(z)$. Then $F_0(z)$ is Cauchy–Riemann holomorphic in $\mathfrak{D}$.*

Proof Let a be an arbitrary point in $\mathfrak{D}$ and let the closed polycylinder $\overline{\Delta(a; r)} \equiv \{z \mid |z_k - a_k| \leq r_k;\, k = 1, \ldots, n\} \subset \mathfrak{D}$. Since for each n $F_n(z)$ is a holomorphic function in the Cauchy–Riemann sense we have for $z \in \Delta(a;r)$ that

$$F_m(z) = \left(\frac{1}{2\pi i}\right)^n \int_{\mathfrak{S}_n} \frac{F_m(\zeta)\, d\zeta_1 \cdots d\zeta_n}{\prod_{k=1}^{n} (\zeta_k - z_k)},$$

where $\mathfrak{S}_n \equiv \prod_{k=1}^{n} \{\zeta \mid |\zeta_k - a_k| = r_k\}$ is the skeleton of $\Delta(a; r)$. In that the partial sums $F_m(z)$ converge uniformly, as $m \to \infty$, to $F_0(z)$ for $\overline{\Delta(a; r)} \subset \mathfrak{D}$ we may pass to the limit under the integral sign, yielding

$$F_0(z) = \lim_{m \to \infty} \left(\frac{1}{2\pi i}\right)^n \int_{\mathfrak{S}_n} \frac{F_m(\zeta)\, d\zeta_1 \cdots d\zeta_n}{\prod_{k=1}^{n} (\zeta_k - z_k)}$$

$$= \left(\frac{1}{2\pi i}\right)^n \int_{\mathfrak{S}_n} \frac{F_0(\zeta)\, d\zeta_1 \cdots d\zeta_n}{\prod_{k=1}^{n} (\zeta_k - z_k)}.$$

We conclude from this that $F_0(z)$ is holomorphic in $\Delta(a; \rho)$, with $\rho_k < r_k$ $(k = 1, \ldots, n)$. Since any compact subset of $\mathfrak{D}$ can be covered by a finite number of polycylinders of the type $\Delta(a; r)$ we conclude that $F_0(z)$ is holomorphic in $\mathfrak{D}$.

From what has been said earlier it is clear that a function holomorphic in the Weierstrass sense at the point $z \in \mathfrak{D}$ is also holomorphic in the Cauchy–Riemann sense. It is easy to show that a function holomorphic in the Cauchy–Riemann sense and *continuous* (in all the variables) in a region $\mathfrak{D} \subset \mathbb{C}^n$ is also Weierstrass holomorphic. To show that this is also the case when we remove the condition of continuity is considerably more difficult, and we postpone this problem for somewhat later.

Theorem 1.1.3 *Let $f(z)$ be Cauchy–Riemann holomorphic and continuous (in all the variables) in the region $\mathfrak{D} \subset \mathbb{C}^n$. Then $f(z)$ is also holomorphic in the sense of Weierstrass. Furthermore, if $f(z)$ is Weierstrass holomorphic in $\mathfrak{D}$ then it is continuous (in all the variables) and Cauchy–Riemann holomorphic in $\mathfrak{D}$.*

Proof We prove this result for $n = 2$, the case of n variables follows by induction. If $a \in \mathfrak{D}$ there then exists a closed polycylinder $\overline{\Delta(a; r)} \subset \mathfrak{D}$ such that by Theorem 1.1.1 we have for $z \in \Delta(a; r)$

$$f(z) = \left(\frac{1}{2\pi i}\right)^2 \int_{\mathfrak{S}_2} \frac{f(\zeta)\, d\zeta}{(\zeta_1 - z_1)(\zeta_2 - z_2)},$$

where $d\zeta = d\zeta_1\, d\zeta_2$ and $\mathfrak{S}_2$ is the skeleton of $\Delta(a; \rho)$. Since

$$\frac{1}{(\zeta_1 - z_1)(\zeta_2 - z_2)} = \sum_{l, k=0}^{\infty} \frac{(z_1 - a_1)^l (z_2 - a_2)^k}{(\zeta_1 - a_1)^{l+1}(\zeta_2 - a_2)^{k+1}}$$

converges uniformly and absolutely for $|z_k - a_k| \leq \rho_k < r_k$ $k = 1, 2$, and $f(\zeta)$ is continuous in $\mathfrak{D}$, then we may multiply this series by $f(\zeta)$ and integrate termwise. We obtain

$$f(z) = \sum_{l, k=0}^{\infty} c_{lk}(z_1 - a_1)^l (z_2 - a_2)^k,$$

where

$$c_{lk} \equiv \left(\frac{1}{2\pi i}\right)^2 \int_{\mathfrak{S}_2} \frac{f(\zeta)\, d\zeta}{(\zeta_1 - a_1)^{l+1}(\zeta_2 - a_2)^{k+1}}. \tag{1.1.7}$$

Furthermore, this series clearly converges uniformly in $\overline{\Delta(a; \rho)}$.

That $f(z)$ is Cauchy–Riemann holomorphic in $\mathfrak{D}$ if it is Weierstrass holomorphic follows, as remarked before, from the fact that we may differentiate a uniformly convergent series termwise in each variable separately. The fact that it must be continuous in all the variables we show as follows.

Proof Let $f(z)$ be expressed as its (m_1, m_2) partial sum plus a remainder term, i.e.,

$$f(z) = \sum_{k,l=0}^{m_1,m_2} c_{kl}(z_1 - a_1)^k (z_2 - a_2)^l + R_{m_1,m_2}(z),$$

and let us consider the difference $f(z+h) - f(z)$. We have then the estimate

$$|f(z+h) - f(z)| \leq \left| \sum_{k,l=0}^{m_1,m_2} c_{kl}[(z_1 + h_1 - a_1)^k (z_2 + h_2 - a_2)^l - (z_1 - a_1)^k (z_2 - a_2)^l] \right| + |R_{m_1,m_2}(z+h) - R_{m_1,m_2}(z)|.$$

For a given $\varepsilon > 0$ we may choose indices (N_1, N_2) such that

$$|R_{m_1,m_2}(z+h)| < \varepsilon/3, \; |R_{m_1,m_2}(z)| < \varepsilon/3 \quad \text{when} \quad m_k > N_k \; (k = 1, 2)$$

and h is sufficiently small. Clearly the (m_1, m_2) partial sum is continuous (as may be seen below) and hence for $\|h\| = (|h_1|^2 + |h_2|^2)^{1/2} < \delta(\varepsilon)$ we have

$$\left| \sum_{k,l=0}^{m_1,m_2} c_{kl}[(z_1 + h_1 - a_1)^k (z_2 + h_2 - a_2)^l - (z_1 - a_1)^k (z_2 - a_2)^l] \right|$$
$$\leq \sum_{k,l=0}^{m_1,m_2} |c_{kl}| \left\{ \sum_{\substack{\mu,\nu=0 \\ \mu+\nu\neq 0}}^{k,l} |h_1|^\mu \cdot |h_2|^\nu \cdot |z_1 - a_1|^{k-\mu} \cdot |z_2 - a_2|^{l-\nu} \binom{k}{\mu} \binom{l}{\nu} \right\} < \frac{\varepsilon}{3}.$$

We conclude that $|f(z+h) - f(z)| < \varepsilon$, and hence Weierstrass holomorphic, is equivalent to Cauchy–Riemann holomorphic plus continuity (in all the variables), which is the desired result.

We remark that in what follows we shall develop the local theory of several complex variables for the case $n = 2$; most of our results carry over immediately to the case $n > 2$ by induction.

Let us suppose the function $f(z)$ is Weierstrass holomorphic in the domain $\mathfrak{D}$; then, about each point $a \in \mathfrak{D}$, $f(z)$ has a power series expansion of the form (1.1.1), which converges in a bicylindrical neighborhood. Considered as a power series in, say, just the variable z_1, for $z_2 = a_2$, the function is clearly analytic and hence its partial derivatives with respect to z_1 may be computed by differentiating the series termwise. Similarly, we may compute the partial derivatives with respect to z_2. Indeed, the derived series are also holomorphic in the two complex variables z_1 and z_2, which may be seen by using the method of dominants. We consider the following general series obtained by formally differentiating termwise with respect to z_1 and z_2:

$$\frac{1}{m!n!} \frac{\partial^{m+n} f(z)}{\partial z_1{}^m \, \partial z_2{}^n} = \sum_{l=m}^{\infty} \sum_{k=n}^{\infty} \binom{l}{m} \binom{k}{n} c_{lk} (z_1 - a_1)^{l-m} (z_2 - a_2)^{k-n} \qquad (1.1.8)$$

Clearly one has from Eq. (1.1.7), the two-variable Cauchy estimates for the coefficient of (1.1.8), i.e.,

$$|c_{lk}| \leq M\rho_1^{-1}\rho_2^{-k}, \tag{1.1.9}$$

where $\rho_k < r_k$, and the series for $f(z)$ converges in the bicylinder $\Delta(a; r)$, $r = (r_1, r_2)$; here $M = M(\rho)$ is the maximum modulus of $f(z)$ on the skeleton of the bicylinder $\Delta(a; \rho)$. Using (1.1.9) we obtain the following estimate,

$$\begin{aligned}
&\frac{1}{m!n!}\left|\frac{\partial^{m+n}f(z)}{\partial z_1{}^m\,\partial z_2{}^n}\right| \\
&\leq \frac{M}{\rho_1{}^m\rho_2{}^n}\left\{\sum_{l=m}^{\infty}\sum_{k=n}^{\infty}\binom{l}{m}\binom{k}{n}\left(\frac{|z_1 - a_1|}{\rho_1}\right)^{l-m}\left(\frac{|z_2 - a_2|}{\rho_2}\right)^{k-n}\right. \\
&= \frac{M}{\rho_1{}^m\rho_2{}^n}\left(1 - \frac{|z_1 - a_1|}{\rho_1}\right)^{-m-1}\left(1 - \frac{|z_2 - a_2|}{\rho_2}\right)^{-n-1},
\end{aligned}$$

from which it follows that the derived series for $\partial^{m+n}f/\partial z_1{}^m\partial z_2{}^n$ is holomorphic at $z = a$. Since this holds for all points $a \in \mathfrak{D}$ we obtain that the derived series is holomorphic in $\mathfrak{D}$. By induction one then has:

Theorem 1.1.4 *If $f(z)$ is Weierstrass holomorphic in the domain $\mathfrak{D}$, then its partial derivatives of all orders are Weierstrass holomorphic in $\mathfrak{D}$.*

We have already observed that if $f(z)$ is Weierstrass holomorphic in $\mathfrak{D}$ then it is also Cauchy–Riemann holomorphic, and the coefficients of the series (1.1.1) are given by (1.1.7). Comparing this with the expression (1.1.8) yields the well-known relationships between the Taylor coefficients and the partial derivatives:

$$\frac{1}{m!n!}\frac{\partial^{m+n}f(a)}{\partial z_1{}^m\,\partial z_2{}^n} = c_{mn}, \tag{1.1.10}$$

and

$$f(z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!n!}\frac{\partial^{m+n}f(a)}{\partial z_1{}^m\,\partial z_2{}^n}(z_1 - a_1)^m(z_2 - a_2)^n. \tag{1.1.11}$$

Let us suppose that the series (1.1.11) converges in the bicylinder $\Delta_r(a) \equiv \{z \,||z_k - a_k| < r_k; k = 1, 2\}$, and consider the formal power series

$$\tilde{f}(z) \equiv \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!n!}\frac{\partial^{m+n}f(z^0)}{\partial z_1{}^m\,\partial z_2{}^n}(z_1 - z_1{}^0)^m(z_2 - z_2{}^0)^n, \tag{1.1.12}$$

where $(z_1{}^0, z_2{}^0) \in \Delta_r(a)$. We shall show that the series for $\tilde{f}(z)$ converges in the bicylinder $\Delta_\rho(z^0)$, where $\rho_k = r_k - |z_k{}^0 - a_k|$, and furthermore in this region $\tilde{f}(z) \equiv f(z)$. It follows then that $f(z)$ as defined by (1.1.11) is Weierstrass holomorphic in the interior of $\Delta_r(a)$.

From (1.1.8) we have

$$\frac{\partial^{m+n} f(z^0)}{\partial z_1{}^m \, \partial z_2{}^n} = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(l+m)!}{l!} \frac{(k+n)!}{k!} c_{l+m,\,k+n} (z_1{}^0 - a_1)^l (z_2{}^0 - a_2)^k,$$

and hence we have

$$\left| \frac{\partial^{m+n} f(z^0)}{\partial z_1{}^m \, \partial z_2{}^n} \right| \leq \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(l+m)!}{l!} \frac{(k+n)!}{k!} |c_{l+m,\,k+n}| \, |z_1{}^0 - a_1|^l \, |z_2{}^0 - a_2|^k.$$

Consequently, one has for an estimate on $\tilde{f}(z)$, when $|z_k - z_k{}^0| \leq \tilde{\rho}_k < \rho_k$, $\rho_k = r_k - |z_k{}^0 - a_k| \qquad (k = 1, 2)$,

$$\begin{aligned}
|\tilde{f}(z)| &\leq \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!} \left| \frac{\partial^{m+n} f(z^0)}{\partial z_1{}^m \, \partial z_2{}^n} \right| |z_1 - z_1{}^0|^m \, |z_2 - z_2{}^0|^n \\
&\leq \sum_{m,n=0}^{\infty} \frac{\tilde{\rho}_1{}^m \tilde{\rho}_2{}^n}{m!n!} \Bigg(\sum_{l,k=0}^{\infty} \frac{(l+m)!(k+n)!}{l!k!} \\
&\qquad \times |c_{l+m,\,k+n}| \, |z_1{}^0 - a_1|^l \, |z_2{}^0 - a_2|^k \Bigg) \\
&\leq \sum_{p,q=0}^{\infty} |c_{p,q}| \left(\sum_{l=0}^{p} \sum_{k=0}^{q} \frac{p!q! \tilde{\rho}_1^{p-l} |z_1{}^0 - a_1|^l \tilde{\rho}_2^{q-k} |z_2{}^0 - a_2|^k}{(p-l)!l!(q-k)!k!} \right) \\
&\leq \sum_{p,q=0}^{\infty} |c_{p,q}| \, \tilde{r}_1{}^p \tilde{r}_2{}^q < \infty.
\end{aligned}$$

Since $\tilde{r}_k \equiv \tilde{\rho}_k + |z_k{}^0 - a_k| < r_k$ $(k = 1, 2)$, the series (1.1.1) with

$$c_{p,q} = \frac{1}{p!q!} \frac{\partial^{p+q} f(a)}{\partial z_1{}^p \, \partial z_2{}^q}$$

is uniformly and absolutely convergent in the bicylinder, $\Delta_{\tilde{r}}(a) \subset \Delta_r(a)$. We realize from this that the series (1.1.12) converges absolutely in $\Delta_{\tilde{\rho}}(z^0)$ and hence we may sum this series by regrouping the terms in various ways. For instance one such grouping gives us

$$\tilde{f}(z) = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \frac{\partial^{m+n} f(z^0)}{\partial z_1{}^m \, \partial z_2{}^n} (z_1 - z_1{}^0)^m (z_2 - z_2{}^0)^n$$

$$= \sum_{m,n=0}^{\infty} \frac{1}{m!n!}(z_1 - z_1^0)^m (z_2 - z_2^0)^n$$

$$\cdot \left(\sum_{l,k=0}^{\infty} \frac{(l+m)!(k+n)!}{l!k!} c_{l+m,k+n}(z_1^0 - a_1)^l (z_2^0 - a_2)^k \right)$$

$$= \sum_{p,q=0}^{\infty} c_{p,q}$$

$$\times \left(\sum_{l=0}^{p} \sum_{k=0}^{q} \frac{p!q!(z_1 - z_1^0)^{p-l}(z_1^0 - a_1)^l (z_2 - z_2^0)^{q-k}(z_2^0 - a_2)^k}{(p-l)!l!(q-k)!k!} \right)$$

$$= \sum_{p,q=0}^{\infty} c_{p,q}[(z_1 - z_1^0) + (z_1^0 - a_1)]^p [(z_2 - z_2^0) + (z_2^0 - a_2)]^q$$

$$= f(z).$$

We summarize the above discussion by the following theorem.

Theorem 1.1.5 *Let $f(z)$ be Weierstrass holomorphic at the point $a \in \mathfrak{D}$ and be represented there by the power series* (1.1.1), *which converges in the bicylinder $\Delta_r(a) \subset \mathfrak{D}$. Then $f(z)$ is Weierstrass holomorphic at each point $z^0 \in \Delta_r(a)$, and has a power series representation, which converges in the bicylinder $\Delta_\rho(z^0)$, where $\rho_k = r_k - |z_k^0 - a_k|$ $(k = 1, 2)$.*

The previous theorem tells us that the regrouped series (1.1.12) must converge at least in the original bicylinder. If on the other hand this series converges in a larger bicylinder, $\Delta_{\rho'}(z^0)$, i.e., where $\rho_k' > r_k - |z_k^0 - a_k|$, this regrouped series serves to provide a direct holomorphic continuation of the function element $(f(z), z^0)$. Choosing a point $z' \in \Delta_{\rho'}(z^0)$ we may again regroup terms of this series about the center z', and if its bicylinder of convergence extends past the boundary of $\Delta_{\rho'}(z^0)$ we have again obtained a continuation of our original function element. Indeed, we shall refer (as in the case of one complex variable) to any function element obtained by a finite chain of direct holomorphic continuations (using bicylinders) as a holomorphic continuation of the original function element.

Let us now define as the *real environment* [B.M. 1, p. 34] of a point $z^0 \in \mathfrak{D}$, any point set containing the rectangle

$$\mathfrak{r} \equiv \{z \mid |x_k - x_k^0| < d;\ y_k = y_k^0;\ k = 1, 2\}.$$

We note that since the partial derivatives $\partial^{m+n} f / \partial z_1^m\, \partial z_2^n$ may be evaluated at $z = z^0$ by just using points of $\mathfrak{r}$, that if $f(z) = 0$ for z in $\mathfrak{r}$, then $f(z) \equiv 0$ in a full neighborhood of z^0. Now if $f(z)$ is given to be holomorphic in the domain $\mathfrak{D}$, then it follows that $f(z) \equiv 0$ in $\mathfrak{D}$, since the value of the function $f(z)$ at each point of $\mathfrak{D}$ may be found with a finite chain of direct holomorphic continuations by bicylinders. From this fact it follows immediately that if two functions, $f_1(z)$ and $f_2(z)$, which are defined in the domains $\mathfrak{D}_1$ and $\mathfrak{D}_2$, respectively, coincide on a real environment of a point $z^0 \in \mathfrak{D}_1 \cap \mathfrak{D}_2$, then there exists a unique function defined in $\mathfrak{D}_1 \cup \mathfrak{D}_2$, which coincides with each of the $f_k(z)$ $(k = 1, 2)$ in their respective domains of definition.†

We are now able, using the information above, to give a precise meaning to the complex form of the Cauchy–Riemann equations. For instance, let us suppose the function $f(z)$ is Weierstrass holomorphic in the domain $\mathfrak{D}$. Then for each point $z^0 \in \mathfrak{D}$ there exists a bicylinder $\Delta_r(z^0)$ such that the power series

$$f(z) = \sum_{l,k=0}^{\infty} c_{lk}(x_1 + iy_1 - z_1{}^0)^l (x_2 + iy_2 - z_2{}^0)^k$$

converges for each $(x_1 + iy_1, x_2 + iy_2) \in \Delta_r(z^0)$. Indeed this series is seen to converge for complex values of x_k, and y_k also, provided that $|x_k| < r_k/2$, $|y_k| < r_k/2$ $(k = 1, 2)$. Hence regrouping the series in terms of powers of x_1, y_1, x_2, y_2, we see that it represents a Weierstrass holomorphic function of these four complex variables in the polycylinder $\Delta_{r/2}^{(4)}(x^0, y^0)$. If we now introduce the linear transformation $z_k = x_k + iy_k$, $z_k{}^* = x_k - iy_k$ $(k = 1, 2)$, the composite function is certainly Weierstrass holomorphic in at least the polycylinder,

$$\Delta_{r/4}^{(4)}(z^0, z^{*0}) \equiv \{(z, z^*) \,|\, |z_k - z_k{}^0| < r_k/4,\ |z_k{}^* - z_k^{0*}| < r_k/4;\ k = 1, 2\}.$$

If the closure of $\mathfrak{D}$ is compact in $\mathbb{C}^2$, then $\mathfrak{D}$ has a finite covering with bicylinders $\Delta_r(z^{(n)})$, each suitably chosen for direct holomorphic continuation of $f(z)$ between overlapping bicylinders. We conclude that in the space of four complex variables, $(x_1, x_2, y_1, y_2) \in \mathbb{C}^4$, the function $\Psi(x, y) \equiv f(z)$ is holomorphic in a four-complex dimensional neighborhood of $\mathfrak{D}$, $\mathcal{N}^{(4)}(\mathfrak{D})$. Likewise the composite function, $\Phi(z, z^*) = \Psi(x, y)$ (obtained by the linear mapping above), and the derived functions $\partial\Phi/\partial z_k{}^*$ $(k = 1, 2)$, are also holomorphic in $\mathcal{N}^{(4)}(\mathfrak{D})$.

Now if as we have assumed, $f(z)$ is holomorphic in $\mathfrak{D}$, then for each point $z^0 \in \mathfrak{D}$, the Weierstrass holomorphic function $\partial\Phi(z, z^*)/\partial z_k{}^*$ $(k = 1, 2)$, defined in the polydisk $\Delta_{r/4}^{(4)}(z^0, z^{*0})$ by the regrouped series, converges there identically to zero. We conclude from this that $\partial\Phi/\partial z_k{}^* \equiv 0$ for $(z, z^*) \in \mathcal{N}^{(4)}(\mathfrak{D})$, and hence in the restriction, $\bar{z}_k = z_k{}^*$ $(k = 1, 2)$, (i.e., x_k and y_k are

† For further results of this kind the reader is referred to [B.M.1, Chapter II].

real), $\partial\Phi/\partial\bar{z}_k \equiv 0$ ($k = 1, 2$). Hence, if we assume $f(z)$ is Weierstrass holomorphic, the complex forms of the Cauchy–Riemann equations have a clearly understood meaning.

2. *Hartogs' Theorem and Holomorphic Continuation*

At this point we are ready to demonstrate that Cauchy–Riemann and Weierstrass holomorphic are equivalent concepts. Afterwards we shall just refer to functions being simply holomorphic. To this end we first prove a theorem known as Hartogs' lemma.

Theorem 1.2.1 (Hartogs' Lemma) *Let $f(z)$ be Cauchy–Riemann holomorphic in the closed bicylinder $\Delta \equiv \{z \,||z_k| \le r_k;\ k = 1, 2\}$, and bounded in the closed bicylinder, $\tilde{\Delta} \equiv \{z \,||z_1| \le r_1,\ |z_2| \le \rho < r_2\}$. Then $f(z)$ is a continuous function of z_1 and z_2 simultaneously for $z \in \Delta$.*

Proof Since $f(z)$ is Cauchy–Riemann holomorphic it is holomorphic in each variable separately, and (since for one complex variable Cauchy–Riemann and Weierstrass holomorphic are obviously equivalent) we have that the series

$$f(z) = \sum_{k=0}^{\infty} f_k(z_1) z_2{}^k \tag{1.2.1}$$

converges uniformly for z_2 such that $|z_2| \le r_2$. Here the variable z_1 is arbitrary with $|z_1| \le r_1$; we remark that it is not self-evident at this time that the functions $f_k(z_1)$ are analytic in $|z_1| \le r_1$. In order to see this we proceed as follows: first, $f(z_1, 0) \equiv f_0(z_1)$ must be holomorphic in $|z_1| \le r_1$; second, so are the functions $F^{(n)}(z_1, z_2)$ (for each fixed z_2, $0 < |z_2| \leqslant r_2$) defined recursively by

$$F^{(n)}(z_1, z_2) = \frac{f(z_1, z_2) - \sum_{k=0}^{n-1} f_k(z_1) z_2{}^k}{z_2{}^n} = \sum_{l=0}^{\infty} f_{l+n}(z_1) z_2{}^l. \tag{1.2.2}$$

Evidently, the function $F^{(n)}(z_1, z_2)$ is Cauchy–Riemann holomorphic in the bicylinder $\{0 < |z_2| < r_2\} \times \{|z_1| < r_1\}$. Let $\{z_2^{(i)}\}$, $i \in I$ (a suitable index set), be a sequence of points in $\{0 < |z_2| < r_2\}$ which converge to $z_2 = 0$. Then