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Vidar Thomée

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Galerkin Finite Element Methods for Parabolic Problems

Second Edition

抛物问题的伽辽金有限元方法 第2版

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Preface

My purpose in this monograph is to present an essentially self-contained account of the mathematical theory of Galerkin finite element methods as applied to parabolic partial differential equations. The emphases and selection of topics reflects my own involvement in the field over the past 25 years, and my ambition has been to stress ideas and methods of analysis rather than to describe the most general and farreaching results possible. Since the formulation and analysis of Galerkin finite element methods for parabolic problems are generally based on ideas and results from the corresponding theory for stationary elliptic problems, such material is often included in the presentation.

The basis of this work is my earlier text entitled *Galerkin Finite Element Methods for Parabolic Problems,* Springer Lecture Notes in Mathematics, No. 1054, from 1984. This has been out of print for several years, and I have felt a need and been encouraged by colleagues and friends to publish an updated version. In doing so I have included most of the contents of the 14 chapters of the earlier work in an updated and revised form, and added four new chapters, on semigroup methods, on multistep schemes, on incomplete iterative solution of the linear algebraic systems at the time levels, and on semilinear equations. The old chapters on fully discrete methods have been reworked by first treating the time discretization of an abstract differential equation in a Hilbert space setting, and the chapter on the discontinuous Galerkin method has been completely rewritten.

The following is an outline of the contents of the book:

In the introductory Chapter 1 we begin with a review of standard material on the finite element method for Dirichlet's problem for Poisson's equation in a bounded domain, and consider then the simplest Galerkin finite element methods for the corresponding initial-boundary value problem for the linear heat equation. The discrete methods are based on associated weak, or variational, formulations of the problems and employ first piecewise linear and then more general approximating functions which vanish on the boundary of the domain. For these model problems we demonstrate the basic error estimates in energy and mean square norms, in the parabolic case first for the semidiscrete problem resulting from discretization in the spatial variables only, and then also for the most commonly used fully discrete schemes obtained by discretization in both space and time, such as the backward Euler and Crank-Nicolson methods.

In the following five chapters we study several extensions and generalizations of the results obtained in the introduction in the case of the spatially semidiscrete approximation, and show error estimates in a variety of norms. First, in Chapter 2, we formulate the semidiscrete problem in terms of a more general approximate solution operator for the elliptic problem in a manner which does not require the approximating functions to satisfy the homogeneous boundary conditions. As an example of such a method we discuss a method of Nitsche based on a nonstandard weak formulation. In Chapter 3 more precise results are shown in the case of the homogeneous heat equation. These results are expressed in terms of certain function spaces $\dot{H}^s(\Omega)$ which are characterized by both smoothness and boundary behavior of its elements, and which will be used repeatedly in the rest of the book. We also demonstrate that the smoothing property for positive time of the solution operator of the initial value problem has an analogue in the semidiscrete situation, and use this to show that the finite element solution converges to full order even when the initial data are nonsmooth. The results of Chapters 2 and 3 are extended to more general linear parabolic equations in Chapter 4. Chapter 5 is devoted to the derivation of stability and error bounds with respect to the maximum-norm for our plane model problem, and in Chapter 6 negative norm error estimates of higher order are derived, together with related results concerning superconvergence.

In the next six chapters we consider fully discrete methods obtained by discretization in time of the spatially semidiscrete problem. First, in Chapter 7, we study the homogeneous heat equation and give analogues of our previous results both for smooth and for nonsmooth data. The methods used for time discretization are of one-step type and rely on rational approximations of the exponential, allowing the standard Euler and Crank-Nicolson procedures as special cases. Our approach here is to first discretize a parabolic equation in an abstract Hilbert space framework with respect to time, and then to apply the results obtained to the spatially semidiscrete problem. The analysis uses eigenfunction expansions related to the elliptic operator occurring in the parabolic equation, which we assume positive definite. In Chapter 8 we generalize the above abstract considerations to a Banach space setting and allow a more general parabolic equation, which we now analyze using the Dunford-Taylor spectral representation. The time discretization is interpreted as a rational approximation of the semigroup generated by the elliptic operator, i.e., the solution operator of the initial-value problem for the homogeneous equation. Application to maximum-norm estimates is discussed. In Chapter 9 we study fully discrete one-step methods for the inhomogeneous heat equation in which the forcing term is evaluated at a fixed finite number of points per time stepping interval. In Chapter 10 we apply Galerkin's method also for the time discretization and seek discrete solutions as piecewise polynomials in the time variable which may be discontinuous at the now not necessarily equidistant nodes. In this *discontinuous Galerkin* procedure the forcing term enters in integrated form rather than at a finite number of points. In Chapter 11 we consider multistep backward difference methods. We first study such methods with constant time steps of order at most 6, and show stability as well as smooth and nonsmooth data error estimates, and then discuss the second order backward difference method with variable time steps. In Chapter 12 we study the incomplete iterative solution of the finite dimensional linear systems of algebraic equations which need to be solved at each level of the time stepping procedure, and exemplify by the use of a V-cycle multigrid algorithm.

The next two chapters are devoted to nonlinear problems. In Chapter 13 we discuss the application of the standard Galerkin method to a model nonlinear parabolic equation. We show error estimates for the spatially semidiscrete problem as well as the fully discrete backward Euler and Crank-Nicolson methods, using piecewise linear finite elements, and then pay special attention to the formulation and analysis of time stepping procedures based on these, which are linear in the unknown functions. In Chapter 14 we derive various results in the case of semilinear equations, in particular concerning the extension of the analysis for nonsmooth initial data from the case of linear homogenous equations.

In the last four chapters we consider various modifications of the standard Galerkin finite element method. In Chapter 15 we analyze the so called lumped mass method for which in certain cases a maximum-principle is valid. In Chapter 16 we discuss the H^1 and H^{-1} methods. In the first of these, the Galerkin method is based on a weak formulation with respect to an inner product in H^1 and for the second, the method uses trial and test functions from different finite dimensional spaces. In Chapter 17, the approximation scheme is based on a mixed formulation of the initial boundary value problem in which the solution and its gradient are sought independently in different spaces. In the final Chapter 18 we consider a singular problem obtained by introducing polar coordinates in a spherically symmetric problem in a ball in $\mathbf{R}^3$ and discuss Galerkin methods based on two different weak formulations defined by two different inner products.

References to the literature where the reader may find more complete treatments of the different topics, and some historical comments, are given at the end of each chapter.

A desirable mathematical background for reading the text includes standard basic partial differential equations and functional analysis, including Sobolev spaces; for the convenience of the reader we often give references to the literature concerning such matters.

The work presented, first in the Lecture Notes and now in this monograph, has grown from courses, lecture series, summer-schools, and written material that I have been involved in over a long period of time. I wish to thank my

students and colleagues in these various contexts for the inspiration and support they have provided, and for the help they have given me as discussion partners and critics. As regards this new version of my work I particularly address my thanks to Georgios Akrivis, Stig Larsson, and Per-Gunnar Martinsson, who have read the manuscript in various degrees of detail and are responsible for many improvements. I also want to express my special gratitude to Yumi Karlsson who typed a first version of the text from the old lecture notes, and to Gunnar Ekolin who generously furnished me with expert help with the intricacies of TEX.

Göteborg
July 1997

Vidar Thomée

Preface to the Second Edition

I am pleased to have been given the opportunity to prepare a second edition of this book. In doing so, I have kept most of the text essentially unchanged, but after correcting a number or typographical errors and other minor inadequacies, I have also taken advantage of this possibility to include some new material representing work that I have been involved in since the time when the original version appeared about eight years ago.

This concerns in particular progress in the application of semigroup theory to stability and error analysis. Using the theory of analytic semigroups it is convenient to reformulate the stability and smoothing properties as estimates for the resolvent of the associated elliptic operator and its discrete analogue. This is particularly useful in deriving maximum-norm estimates, and has led to improvements for both spatially semidiscrete and fully discrete problems. For this reason a somewhat expanded review of analytic semigroups is given in the present Chapter 6, on maximum-norm estimates for the semidiscrete problem, where now resolvent estimates for piecewise linear finite elements are discussed in some detail. These changes have affected the chapter on single step time stepping methods, expressed as rational approximation of semigroups, now placed as Chapter 9. The new emphasis has led to certain modifications and additions also in other chapters, particularly in Chapter 10 on multistep methods and Chapter 15 on the lumped mass method.

I have also added two chapter at the end of the book on other topics of recent interest to me. The first of these, Chapter 19, concern problems in which the spatial domain is polygonal, with particular attention given to noncovex such domains. rather than with smooth boundary, as in most of the rest of the book. In this case the corners generate singularites in the exact solution, and we study the effect of these on the convergence of the finite element solution.

The second new chapter, Chapter 20, considers an alternative to time stepping as a method for discretization in time, which is based on representing the solution as an integral involving the resolvent of the elliptic operator along a smooth curve extending into the right half of the complex plane, and then applying an accurate quadrature rule to this integral. This reduces the parabolic problem to a finite set of elliptic problems that may be solved in parallel. The method is then combined with finite element discretization in the spatial variable. When applicable, this method gives very accurate approximations of the exact solution in an efficient way.

I would like to take this opportunity to express my warm gratitude to Georgios Akrivis for his generous help and support. He has critically read through the new material and made many valuable suggestions.

Göteborg
March 2006

Vidar Thomée

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1. The Standard Galerkin Method

In this introductory chapter we shall study the standard Galerkin finite element method for the approximate solution of the model initial-boundary value problem for the heat equation,

$$\begin{aligned} (1.1) \qquad u_t - \Delta u &= f \quad \text{in } \Omega, \qquad \text{for } t > 0, \\ u &= 0 \quad \text{on } \partial\Omega, \quad \text{for } t > 0, \quad \text{with } u(\cdot, 0) = v \quad \text{in } \Omega, \end{aligned}$$

where Ω is a domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$, and where $u = u(x,t)$, u_t denotes $\partial u/\partial t$, and $\Delta = \sum_{j=1}^d \partial^2/\partial x_j^2$ the Laplacian.

Before we start to discuss this problem we shall briefly review some basic relevant material about the finite element method for the corresponding stationary problem, the Dirichlet problem for Poisson's equation,

$$(1.2) \qquad -\Delta u = f \quad \text{in } \Omega, \quad \text{with } u = 0 \quad \text{on } \partial\Omega.$$

Using a variational formulation of this problem, we shall define an approximation of the solution u of (1.2) as a function u_h which belongs to a finite-dimensional linear space S_h of functions of x with certain properties. This function, in the simplest case a continuous, piecewise linear function on some partition of Ω, will be a solution of a finite system of linear algebraic equations. We show basic error estimates for this approximate solution in energy and least square norms.

We shall then turn to the parabolic problem (1.1) which we first write in a weak form. We then proceed to discretize this problem, first in the spatial variable x, which results in an approximate solution $u_h(\cdot, t)$ in the finite element space S_h, for $t \geq 0$, as a solution of an initial value problem for a finite-dimensional system of ordinary differential equations. We then define the fully discrete approximation by application of some finite difference time stepping method to this finite dimensional initial value problem. This yields an approximate solution $U = U_h$ of (1.1) which belongs to S_h at discrete time levels. Error estimates will be derived for both the spatially and fully discrete solutions.

For a general $\Omega \subset \mathbb{R}^d$ we denote below by $\| \cdot \|$ the norm in $L_2 = L_2(\Omega)$ and by $\| \cdot \|_r$ that in the Sobolev space $H^r = H^r(\Omega) = W_2^r(\Omega)$, so that for real-valued functions v,

$$\|v\| = \|v\|_{L_2} = \Big(\int_\Omega v^2\,dx\Big)^{1/2},$$

and, for r a positive integer,

$$\|v\|_r = \|v\|_{H^r} = \Big(\sum_{|\alpha|\le r} \|D^\alpha v\|^2\Big)^{1/2}, \tag{1.3}$$

where, with $\alpha = (\alpha_1, \dots, \alpha_d)$, $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_d)^{\alpha_d}$ denotes an arbitrary derivative with respect to x of order $|\alpha| = \sum_{j=1}^d \alpha_j$, so that the sum in (1.3) contains all such derivatives of order at most r. We recall that for functions in $H_0^1 = H_0^1(\Omega)$, i.e., the functions v with $\nabla v = \operatorname{grad} v$ in L_2 and which vanish on $\partial\Omega$, $\|\nabla v\|$ and $\|v\|_1$ are equivalent norms (Friedrichs' lemma, see, e.g., [42] or [51]), and that

$$c\|v\|_1 \le \|\nabla v\| \le \|v\|_1, \quad \forall v \in H_0^1, \qquad \text{with } c > 0. \tag{1.4}$$

Throughout this book c and C will denote positive constants, not necessarily the same at different occurrences, which are independent of the parameters and functions involved.

We shall begin by recalling some basic facts concerning the Dirichlet problem (1.2). We first write this problem in a weak, or variational, form: We multiply the elliptic equation by a smooth function φ which vanishes on $\partial\Omega$ (it suffices to require $\varphi \in H_0^1$), integrate over Ω, and apply Green's formula on the left-hand side, to obtain

$$(\nabla u, \nabla\varphi) = (f, \varphi), \quad \forall \varphi \in H_0^1, \tag{1.5}$$

where we have used the L_2 inner products,

$$(v, w) = \int_\Omega vw\,dx, \quad (\nabla v, \nabla w) = \int_\Omega \sum_{j=1}^d \frac{\partial v}{\partial x_j}\frac{\partial w}{\partial x_j}\,dx. \tag{1.6}$$

A function $u \in H_0^1$ which satisfies (1.5) is called a variational solution of (1.2). It is an easy consequence of the Riesz representation theorem that a unique such solution exists if $f \in H^{-1}$, the dual space of H_0^1. In this case (f, φ) denotes the value of the functional f at φ. Further, since we have assumed the boundary $\partial\Omega$ to be smooth, the solution u is smoother by two derivatives in L_2 than the right hand side f, which may be expressed in the form of the *elliptic regularity* inequality

$$\|u\|_{m+2} \le C\|\Delta u\|_m = C\|f\|_m, \quad \text{for any } m \ge -1. \tag{1.7}$$

In particular, using also Sobolev's embeddning theorem, this shows that the solution u belongs to C^∞ if f belongs to C^∞. We refer to, e.g., Evans [96] for such material.