Topics in Measure Theory and Real Analysis

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Preface

This book is concerned with questions of classical measure theory and related topics of real analysis. At the beginning, it should be said that the choice of material included in the present book was completely dictated by research interests and preferences of the author. Nevertheless, we hope that this material will be of interest to a wide audience of mathematicians and, primarily, to those who are working in various branches of modern mathematical analysis, probability theory, the theory of stochastic processes, general topology, and functional analysis. In addition, we touch upon deep set-theoretical aspects of the topics discussed in the book; consequently, set-theorists may detect nontrivial items of interest to them and find out new applications of set-theoretical methods to various problems of measure theory and real analysis. It should also be noted that questions treated in this book are related to material found in the following three monographs previously published by the author.

1) Transformation Groups and Invariant Measures, World Scientific Publ. Co., London-Singapore, 1998.
2) Nonmeasurable Sets and Functions, North-Holland Mathematics Studies, Elsevier, Amsterdam, 2004.
3) Strange Functions in Real Analysis, 2nd edition, Chapman and Hall/CRC, Boca Raton, 2006.

For the convenience of our readers, we will first, briefly and schematically, describe the scope of this book.

In Chapter 1, we consider the general problem of extending partial real-valued functions which, undoubtedly, is one of the most important problems in all of contemporary mathematics and which deserves to be discussed thoroughly. Since the satisfactory solution to this task requires a separate monograph, we certainly do not intend on entering deeply into various aspects of the problem of extending partial functions, but rather we restrict ourselves to several examples that are important for real analysis and classical measure theory and vividly show the fundamental character of this problem. The corresponding examples are given in Chapter 1 and illustrate different approaches and appropriate research methods. Notice that some of the examples presented in this chapter are considered in more details in subsequent sections of the book.

Chapter 2 is devoted to a special, but very important, case of the extension problem for real-valued partial functions. Namely, we discuss therein several variants of the so-called measure extension problem and we pay our attention to purely set-theoretical, algebraic and topological aspects of this problem. In the same chapter, the classical method of extending measures, developed by Marczewski (see [234] and [235]), is presented. Also, a useful theorem is proved which enables us to extend any σ-finite measure μ on a base set E to a measure μ' on the same E, such that all members of a given family of pairwise disjoint subsets of E become μ'-measurable (see [1] and [13]). This theorem is then repeatedly applied in further sections of the book.

In Chapters 3 and 4 we primarily deal with those measures on E which are invariant or quasi-invariant with respect to a certain group of transformations of E. It is widely known that invariant and quasi-invariant measures play a central role in the theory of topological groups, functional analysis, and the theory of dynamical systems. We discuss some general properties of invariant and quasi-invariant measures that are helpful in various fields of mathematics. First of all, we mean the existence and uniqueness properties of such measures. The problem of the existence and uniqueness of an invariant measure naturally arises for a locally compact topological group endowed with the group of all its left (right) translations. In this way, we come to the well-known Haar measure. The theory of Haar measure is thoroughly covered in many text-books and monographs (see, for instance, [80], [83], [182], [202]), so we leave aside the main aspects of this theory. But we present the classical Bogoliubov-Krylov theorem on the existence of a dynamical system for a one-parameter group of homeomorphisms of a compact metric space E. More precisely, we formulate and prove a significant generalization of the Bogoliubov-Krylov statement: the so-called fixed-point theorem of Markov and Kakutani ([93], [168]) for a solvable group of affine continuous transformations of a nonempty compact convex set in a Hausdorff topological vector space. In the same chapters, we distinguish the following two situations: the case when a given topological space E is locally compact and the case when E is not locally compact. The latter case involves the class of all infinite-dimensional Hausdorff topological vector spaces for which the problem of the existence of a nonzero σ-finite invariant (respectively, quasi-invariant) Borel measure needs a specific formulation. Some results in this direction are presented with necessary comments.

Chapter 5 is concerned with measurability properties of real-valued functions defined on an abstract space E, when a certain class M of measures on E is determined. We introduce three notions for a given function f acting from E into the real line $\mathbf{R}$. Namely, f may be

(a) absolutely nonmeasurable with respect to M,
(b) relatively measurable with respect to M,
(c) absolutely (or universally) measurable with respect to M.

We examine these notions and show their close connections with some classical constructions in measure theory. It should be pointed out that the standard concept of measurability of f with respect to a fixed measure μ on E is a particular case of the notions (b) and (c). In this case, the role of M is played by the one-element class $\{\mu\}$.

In Chapter 6 we discuss, again from the measure-theoretical point of view, some properties of the so-called step-functions. Since step-functions are rather simple representatives of the class of all functions (namely, the range of a step-function is at most countable), it is reasonable to consider them in connection with the measure extension problem. It turns out that the behavior of such functions is essentially different in the case of ordinary measures and in the case of invariant (quasi-invariant) measures.

In Chapter 7, we introduce and investigate the class of almost measurable real-valued functions on $\mathbf{R}$. This class properly contains the class of all Lebesgue measurable functions on $\mathbf{R}$ and has certain interesting features. A characterization of almost measurable functions is given and it is shown that any almost measurable function becomes measurable with respect to a suitable extension of the standard Lebesgue measure λ on $\mathbf{R}$.

Chapter 8 focuses on several important facts from general topology. In particular, Kuratowski's theorem (see, for instance, [58], [101], [149]) on closed projections is presented with some of its applications among which we especially examine the existence of a co-meager set of continuous nowhere differentiable functions in the classical Banach space $C[0,1]$. Also, we prove a deep theorem on the existence of Borel selectors for certain partitions of a Polish topological space, which is essentially used in the sequel.

In Chapter 9 the concept of the weak transitivity of an invariant measure is considered and its influence on the existence of nonmeasurable sets is underlined. Here it is vividly shown that some old ideas of Minkowski [173] which were successfully applied by him in convex geometry and geometric number theory, are also helpful in constructions of various paradoxical (e.g., nonmeasurable) sets. Actually, Minkowski had at hand all the needed tools to prove the existence of those subsets of the Euclidean space $\mathbf{R}^n$ $(n \geqslant 1)$, which are nonmeasurable with respect to the classical Lebesgue measure λ_n on $\mathbf{R}^n$.

Chapter 10 covers bad subgroups of an uncountable solvable group $(G, \cdot)$. The term "bad", of course, means the nonmeasurability of a subgroup with respect to a given nonzero σ-finite invariant (quasi-invariant) measure μ on G. We establish the existence of such subgroups of G and, moreover, show that some of them can be applied to obtain invariant (quasi-invariant) extensions of μ. So, despite their bad structural properties, certain nonmeasurable subgroups of G have a positive side from the view-point of the general measure extension problem.

The next two chapters (i.e., Chapters 11-12) are devoted to the structure of algebraic sums of small (in a certain sense) subsets of a given uncountable commutative group $(G, +)$. Recall that the first deep result in this direction was obtained by Sierpiński in his classical work [219] where he stated that there are two Lebesgue measure zero subsets of the real line **R**, whose algebraic (i.e., Minkowski's) sum is not Lebesgue measurable. Let us stress that in [219] the technique of Hamel bases was heavily exploited and in the sequel such an approach became a powerful research tool for further investigations. We develop Sierpiński's above-mentioned result and generalize it in two directions. Namely, we consider the purely algebraic aspect of the problem and its topological aspect as well. The difference between these two aspects is primarily caused by two distinct concepts of "smallness" of subsets of **R**.

In Chapters 13 and 14 we turn our attention to Sierpiński-Zygmund functions [225] and study them from the point of view of the measure extension problem. It is well known that the restriction of a Sierpiński-Zygmund function to any subset of **R** of cardinality continuum is discontinuous. This circumstance directly implies that a Sierpiński-Zygmund function is nonmeasurable in the Lebesgue sense and, moreover, is nonmeasurable with respect to the completion of an arbitrary nonzero σ-finite diffused Borel measure on **R**. In addition, no Sierpiński-Zygmund function has the Baire property.

We give two new constructions of Sierpiński-Zygmund functions.

(1) The construction of a Sierpiński-Zygmund function which is absolutely nonmeasurable with respect to the class of all nonzero σ-finite diffused measures on **R**. (Notice that this result needs some extra set-theoretical axioms.)

(2) The construction of a Sierpiński-Zygmund function which is relatively measurable with respect to the class of all translation-invariant extensions of the Lebesgue measure λ on **R**. (This result does not need any additional set-theoretical hypotheses.)

Chapters 15-17 are similar to each other in the sense that the main topics discussed therein are connected with different constructions of nonseparable extensions of σ-finite measures. Among the results presented in these chapters, let us especially mention:

(i) the construction (assuming the Continuum Hypothesis) of a nonseparable extension of the Lebesgue measure without producing new null-sets;
(ii) the construction (also under some additional set-theoretical axioms) of nonseparable invariant extensions of σ-finite invariant measures by using their nontrivial ergodic components;
(iii) the construction (assuming again the Continuum Hypothesis) of a nonseparable non-atomic left invariant σ-finite measure on any uncountable solvable group.

In Chapter 18, we consider universally measurable additive functionals. The universal measurability is treated here in a generalized sense, namely, a real-valued functional f on a Hilbert space E is universally measurable if and only if for any σ-finite Borel measure μ given on E, there exists an extension μ' of μ such that f becomes measurable with respect to μ'. It is established that there are universally measurable additive functionals which are everywhere discontinuous. This result may be regarded as a counter-version to the well-known statement (see, e.g., [97], [153], [154]), according to which any universally measurable (in the usual sense) additive functional on E is necessarily continuous.

Chapter 19 is devoted to certain strange subsets of the Euclidean plane $\mathbf{R}^2$. We discuss various properties of these paradoxical sets from the measure-theoretical view-point. In particular, a subset Z of $\mathbf{R}^2$ is constructed which is almost invariant under the group of all translations of $\mathbf{R}^2$, is λ_2-thick (where λ_2 stands for the two-dimensional Lebesgue measure on $\mathbf{R}^2$) and, in addition, has the property that for each straight line l in $\mathbf{R}^2$, the intersection $l \cap Z$ is of cardinality strictly less than the cardinality of the continuum. By using this set Z, we define translation-invariant extensions of λ_2 for which no analogue of the classical Fubini theorem can be valid.

The final chapter is connected with certain restrictions of functions acting from **R** into **R**. The first examples of those restrictions of measurable functions, which are defined on sufficiently large subsets of **R** and have various nice properties, were given in widely known statements of real analysis. For instance, in accordance with the classical Luzin theorem (see, e.g., [16], [26], [65], [80], [161], [183], and [192]), every real-valued Lebesgue measurable function restricted to a certain set of strictly positive λ-measure becomes continuous (and an analogous purely topological result holds true in terms of the Baire property and category). We touch upon some other results in this direction. In particular, it is proved that every Lebesgue measurable function $g : \mathbf{R} \to \mathbf{R}$ is monotone on a nonempty perfect subset of $\mathbf{R}$. At the same time, such a perfect set does not need to be of strictly positive λ-measure. The last circumstance is shown by considering Jarnik's [88] continuous nowhere approximately differentiable function whose existence is a rather deep theorem of real analysis (cf. [33], [34], and [167]).

In general, we tried to present the material in a self-contained form completely accessible to graduate and post-graduate students. Moreover, for the reader's convenience, six Appendices are attached to the main text of this book, which can be read independently of the material covered in the chapters.

In Appendix 1 some auxiliary set-theoretical facts and constructions are considered, which are essential in various sections of the book. Namely, elements of infinite combinatorics (e.g., infinite trees and König's lemma), several delicate set-theoretical statements, and the existence of an uncountable universal measure zero subset of $\mathbf{R}$ are discussed.

Appendix 2 is devoted to various theorems on the existence of measurable selectors. Results of this type are important and attractive and have found applications in numerous branches of modern mathematics. We begin with the Choquet theorem on capacities (see [24], [52], [187]) and show its close connection with statements about measurable selectors. Also, we present the following two fundamental results in this topic: the theorem of Kuratowski and Ryll-Nardzewski [151] and, as one of its consequences, the Luzin-Jankov-von Neumann theorem (see, e.g., [99]).

In Appendix 3 deep properties of σ-finite Borel measures on metrizable topological spaces are examined. It is proved that if the topological weight of a metric space E is not measurable in the Ulam sense, then any σ-finite Borel measure on E admits a separable support (cf. [192]). This important fact is essentially used in Chapter 12.

Appendix 4 contains a detailed proof of the existence of a continuous function $f : \mathbf{R} \to \mathbf{R}$ which is nowhere approximately differentiable. As mentioned above, the first example of such a function was constructed by Jarnik in his remarkable work [88]. A function of this type is needed for considerations in Chapter 20.

In Appendix 5 general properties of commutative groups (primarily, infinite commutative groups) are examined and one useful theorem, due to Kulikov, on the algebraic structure of such groups is proved. This theorem is utilized in several parts of the main text of the book; see especially Chapters 10 and 11.

Appendix 6 presents elements of classical descriptive set theory. Namely, we touch upon certain properties of Borel and analytic (Suslin) subsets of uncountable Polish spaces and apply those properties to the question of measurability of sets or functions. (In this context, Appendix 2 is also helpful.) The so-called separation principle, first introduced and extensively studied by Luzin and Sierpiński, receives special attention. Of course, our presentation of this material is concise and superficial. The standard monographs or text-books devoted to classical descriptive set theory are [99], [148], [150], [160], and [162]; see also Martin's article in [10].

Finally, we would like to note that all sections of the book, including the Appendices, are provided with exercises which contain additional information concerning the questions under discussion. Some of the exercises are quite easy but some of them involve difficult mathematical facts and need intensive efforts for their solution. These more difficult exercises are marked by the symbol * and we recommend that the reader solve them in order to better understand the subject. Finally, we also state in the text several unsolved problems which are motivated by (or closely connected with) topics presented in this book.

The Bibliography consists of 251 titles and contains only the most relevant ones. Of course, it is far from being complete but rather provides a basic orientation to the subject in order to stimulate further interest of our readers in various questions of measure theory and real analysis.

A.B. Kharazishvili

Contents

Chapter 1

The problem of extending partial functions

There are several general concepts and ideas in contemporary mathematics which play a fundamental role in almost all of its branches. Among ideas of this kind, the concept of extending a given partial function is of undoubted interest and of paramount importance for various domains of mathematics. For instance, every working mathematician knows that the problem of extending partial functions is considered and intensively studied in universal algebra, general and algebraic topology, mathematical and functional analysis, as well as other fields. Of course, this problem has specific features in any of the above-mentioned disciplines and it frequently needs special approaches or appropriate research tools which are suitable only for a given situation and are applicable to concrete mathematical objects, for example, groups, topological spaces, ordered sets, differentiable manifolds, and other structures.

However, this problem can also be examined from the abstract view-point and methodological conclusions of a general character can be made. Below, we touch upon different aspects of the problem and illustrate them by relevant examples. Some of those examples will be envisaged more thoroughly in subsequent sections of this book. The main goal of our preliminary consideration is to demonstrate how the problem of extending partial functions accumulates ideas from different areas of modern mathematics.

The best known example of this type is the famous Tietze-Urysohn theorem which states that every real-valued continuous function defined on a closed subset of a normal topological space $(E, \mathscr{T})$ can be extended to a real-valued continuous function defined on the whole space E (see, for instance, [58], [101], and [148]).

Another example of this kind is the classical Hahn-Banach theorem which states that a continuous linear functional defined on a vector subspace of a given normed vector space $(E, ||\cdot||)$ can be extended to a continuous linear functional having the same norm and defined on the whole E (see any text-book of functional analysis, for instance, [56], [57], or [209]).

The third example of this sort, although far from the main topics of topology and analysis, is the problem of extending a given partially recursive function to a recursive function. As known, the latter should be defined on the set $\mathbf{N}$ $(= \omega)$ of all natural numbers. The existence of a partially recursive function that does not admit an extension to a recursive function is crucial for basic statements of mathematical logic and the theory of algorithms. It suffices to mention Gödel's incompleteness theorem of the formal arithmetic (see, for instance, [10] and [215]).

Obviously, many other interesting and important examples can be pointed out in this context.

The present book contains selected topics of measure theory which is a necessary part of modern mathematical and functional analysis. As is well known, ordinary measures are real-valued functions defined on certain classes of subsets of a given base set E and having the countable additivity property. Of course, in contemporary mathematics the so-called vector-valued measures and operator-valued measures are also extremely important and are used in many questions of analysis and the theory of stochastic processes, but we do not touch them in our further considerations. Here we would like to stress especially that topics presented in this book are primarily concentrated around the measure extension problem which plays a significant role in numerous questions of real analysis, probability theory, and set-theoretical topology. Actually, the measure extension problem will be central for us in most sections of the book.

Consequently, it is reasonable to begin our preliminary discussion by considering several facts from mathematical analysis, which are closely connected to extensions of partial real-valued functions. Some of the facts listed below are fairly standard and accessible to average-level students. But among the presented facts the reader will also encounter those which are more important and deeper and which find applications in various domains of modern mathematics.

Let $\mathbf{R}$ denote the real line and let X be an arbitrary subset of $\mathbf{R}$.

A function $f : X \to \mathbf{R}$ is called a partial function acting from $\mathbf{R}$ into $\mathbf{R}$.

For this f, we may write $f : \mathbf{R} \to \mathbf{R}$ saying that f is a partial function whose domain is contained in $\mathbf{R}$. As usual, we denote

$$\mathrm{dom}(f) = X.$$

If $X = \mathbf{R}$, then we obviously have an ordinary function $f : \mathbf{R} \to \mathbf{R}$.

The symbol $\mathrm{ran}(f)$ denotes the range of a partial function f, i.e.,

$$\mathrm{ran}(f) = \{f(x) : x \in \mathrm{dom}(f)\}.$$

If Y is any subset of $\mathbf{R}$, then the symbol $f|Y$ stands for the restriction of a partial function f to Y.

As a rule, people working in classical mathematical analysis are often interested in the following general question.

Does there exist an extension $f^* : \mathbf{R} \to \mathbf{R}$ of a partial function $f : \mathbf{R} \to \mathbf{R}$ such that f^* is defined on the whole $\mathbf{R}$ and has certain "nice" properties?

In particular, we may require that f^* should be differentiable or continuous or semicontinuous or monotone or convex or Borel measurable or Lebesgue measurable or should have the Baire property. (Notice that the Baire property can be regarded as a topological version of measurability; extensive material about this property is contained in remarkable books [148], [176], and [192].)

An analogous question arises in a more general situation, e.g., for partial functions f acting from subsets of an abstract set E into $\mathbf{R}$, where E is assumed to be endowed with some additional structure. In such a case an extension

$$f^* : E \to \mathbf{R}$$

with $\mathrm{dom}(f^*) = E$ must preserve a given structure on E or should be compatible, in an appropriate sense, with this structure. It is clear that questions of the above-mentioned kind quite frequently arise in mathematical analysis, general topology, and abstract algebra. Therefore, this topic is of interest for large groups of the working mathematicians.

Below, we have made a small list of results in this direction, have commented on each of them or have given a necessary explanation, and have referred the reader to other related works in which extensions of partial functions are considered more thoroughly (see, [58], [70], [83], [101], and [148]).

For the convenience of potential readers, the material below is presented in the form of examples of statements about extensions of real-valued partial functions. By the way, we think that in various lecture courses for students it is useful to provide them with additional information concerning extensions of partial functions. Such an approach essentially helps them to see more vividly deep connections and interactions between distinct fields of contemporary mathematics. In addition, the students should know that the general problem of extending partial functions is important for all mathematics because this problem almost permanently occurs in different mathematical branches and finds numerous applications.