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Elementary Analysis: The Theory of Calculus

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Preface

A study of this book, and especially the exercises, should give the reader a thorough understanding of a few basic concepts in analysis such as continuity, convergence of sequences and series of numbers, and convergence of sequences and series of functions. An ability to read and write proofs will be stressed. A precise knowledge of definitions is essential. The beginner should memorize them; such memorization will help lead to understanding.

Chapter I sets the scene and, except for the completeness axiom, should be more or less familiar. Accordingly, readers and instructors are urged to move quickly through this chapter and refer back to it when necessary. The most critical sections in the book are Sections 7 through 12 in Chapter II. If these sections are thoroughly digested and understood, the remainder of the book should be smooth sailing.

The first four chapters form a unit for a short course on analysis. I cover these four chapters (except for the optional sections and Section 20) in about 38 class periods; this includes time for quizzes and examinations. For such a short course, my philosophy is that the students are relatively comfortable with derivatives and integrals but do not really understand sequences and series, much less sequences and series of functions, so Chapters I–IV focus on these topics. On two or three occasions I draw on the Fundamental Theorem of Calculus or the Mean Value Theorem, which appear later in the book, but of course these important theorems are at least discussed in a standard calculus class.

In the early sections, especially in Chapter II, the proofs are very detailed with careful references for even the most elementary facts. Most sophisticated readers find excessive details and references a hindrance (they break the flow of the proof and tend to obscure the main ideas) and would prefer to check the items mentally as they proceed. Accordingly, in later chapters the proofs will be somewhat less detailed and references for the simplest facts will often be omitted. This should help prepare the reader for more advanced books which frequently give very brief arguments.

Mastery of the basic concepts in this book should make the analysis in such areas as complex variables, differential equations, numerical analysis, and statistics more meaningful. The book can also serve as a foundation for an in depth study of real analysis given in books such as [2], [11], [13], [14], [17], [19], and [20] listed in the bibliography.

Readers planning to teach calculus will also benefit from a careful study of analysis. Even after studying this book (or writing it) it will not be easy to handle questions such as "What is a number?", but at least this book should help give a clearer picture of the subtleties to which such questions lead.

The optional sections contain discussions of some topics that I think are important or interesting. Sometimes the topic is dealt with lightly and suggestions for further reading are given. Though these sections are not particularly designed for classroom use, I hope that some readers will use them to broaden their horizons and see how this material fits in the general scheme of things.

I have benefitted from numerous helpful suggestions from my colleagues Robert Freeman, William Kantor, Richard Koch, and John Leahy, and from Timothy Hall, Gimli Khazad, and Jorge López. I have also had helpful conversations with my wife Lynn concerning grammar and taste. Of course, remaining errors in grammar and mathematics are the responsibility of the author.

Contents

*Optional sections.

CHAPTER I

Introduction

The underlying space for all the analysis in this book is the set of real numbers. In this chapter we set down some basic properties of this set. These properties will serve as our axioms in the sense that it is possible to derive all the properties of the real numbers using only these axioms. However, we will avoid getting bogged down in this endeavor. Some readers may wish to refer to the appendix on set notation.

§1. The Set ℕ of Natural Numbers

We denote the set $\{1, 2, 3, \ldots\}$ of all *natural numbers* by ℕ. Elements of ℕ will also be called *positive integers*. Each natural number n has a successor, namely $n+1$. Thus the successor of 2 is 3, and 37 is the successor of 36. You will probably agree that the following properties of ℕ are obvious; at least the first four are.

N1. 1 belongs to ℕ.
N2. If n belongs to ℕ, then its successor $n+1$ belongs to ℕ.
N3. 1 is not the successor of any element in ℕ.
N4. If n and m in ℕ have the same successor, then $n = m$.
N5. A subset of ℕ which contains 1, and which contains $n+1$ whenever it contains n, must equal ℕ.

Properties N1 through N5 are known as the *Peano Axioms* or *Peano Postulates*. It turns out that all the properties of ℕ can be proved based on these five axioms; see [3] or [15].

Let's focus our attention on axiom N5, the one axiom that may not be obvious. Here is what the axiom is saying. Consider a subset S of $\mathbb{N}$ as described in N5. Then 1 belongs to S. Since S contains $n+1$ whenever it contains n, it follows that S must contain $2=1+1$. Again, since S contains $n+1$ whenever it contains n, it follows that S must contain $3=2+1$. Once again, since S contains $n+1$ whenever it contains n, it follows that S must contain $4=3+1$. We could continue this monotonous line of reasoning to conclude that S contains any number in $\mathbb{N}$. Thus it seems reasonable to conclude that $S=\mathbb{N}$. It is this reasonable conclusion that is asserted by axiom N5.

Here is another way to view axiom N5. Assume axiom N5 is false. Then $\mathbb{N}$ contains a set S such that

(i) $1\in S$,
(ii) if $n\in S$, then $n+1\in S$,

and yet $S\neq\mathbb{N}$. Consider the smallest member of the set $\{n\in\mathbb{N}: n\notin S\}$, call it n_0. Since (i) holds, it is clear that $n_0\neq 1$. So n_0 must be a successor to some number in $\mathbb{N}$, namely n_0-1. We must have $n_0-1\in S$ since n_0 is the smallest member of $\{n\in\mathbb{N}: n\notin S\}$. By (ii), the successor of n_0-1, namely n_0, must also be in S, which is a contradiction. This discussion may be plausible, but we emphasize that we have not *proved* axiom N5 using the successor notion and axioms N1 through N4, because we implicitly used two unproven facts. We assumed that every nonempty subset of $\mathbb{N}$ contains a least element and we assumed that if $n_0\neq 1$ then n_0 is the successor to some number in $\mathbb{N}$.

Axiom N5 is the basis of *mathematical induction*. Let $P_1, P_2, P_3, \ldots$ be a list of statements or propositions that may or may not be true. The principle of mathematical induction asserts that all the statements $P_1, P_2, P_3, \ldots$ are true provided

(I_1) P_1 is true,
(I_2) P_{n+1} is true whenever P_n is true.

We will refer to (I_1), i.e., the fact that P_1 is true, as the *basis for induction* and we will refer to (I_2) as the *induction step*. For a sound proof based on mathematical induction, properties (I_1) and (I_2) must both be verified. In practice, (I_1) will be easy to check.

EXAMPLE 1. Prove $1+2+\cdots+n=\frac{1}{2}n(n+1)$ for natural numbers n.

SOLUTION. Our nth proposition is

$$P_n: \quad \text{“}1+2+\cdots+n=\tfrac{1}{2}n(n+1).\text{”}$$

Thus P_1 asserts that $1=\frac{1}{2}\cdot 1(1+1)$, P_2 asserts that $1+2=\frac{1}{2}\cdot 2(2+1)$, P_{37} asserts that $1+2+\cdots+37=\frac{1}{2}\cdot 37(37+1)=703$, etc. In particular, P_1 is a true assertion which serves as our basis for induction.

For the induction step, suppose that P_n is true. That is, we suppose

$$1+2+\cdots+n=\tfrac{1}{2}n(n+1)$$

is true. Since we wish to prove P_{n+1} from this, we add $n+1$ to both sides to obtain

$$\begin{aligned}1+2+\cdots+n+(n+1)&=\tfrac{1}{2}n(n+1)+(n+1)\\&=\tfrac{1}{2}\left[n(n+1)+2(n+1)\right]=\tfrac{1}{2}(n+1)(n+2)\\&=\tfrac{1}{2}(n+1)((n+1)+1).\end{aligned}$$

Thus P_{n+1} holds if P_n holds. By the principle of mathematical induction, we conclude that P_n is true for all n. □

We emphasize that prior to the last sentence of our solution we did *not* prove "P_{n+1} is true." We merely proved an implication: "if P_n is true, then P_{n+1} is true." In a sense we proved an infinite number of assertions, namely: P_1 is true; if P_1 is true then P_2 is true; if P_2 is true then P_3 is true; if P_3 is true then P_4 is true; etc. Then we applied mathematical induction to conclude P_1 is true, P_2 is true, P_3 is true, P_4 is true, etc. We also confess that formulas like the one just proved are easier to prove than to derive. It can be a tricky matter to guess such a result. Sometimes results such as this are discovered by trial and error.

EXAMPLE 2. All numbers of the form 7^n-2^n are divisible by 5.

SOLUTION. More precisely, we show that 7^n-2^n is divisible by 5 for each $n\in\mathbb{N}$. Our nth proposition is

$$P_n:\ \text{“}7^n-2^n \text{ is divisible by 5.”}$$

The basis for induction P_1 is clearly true, since $7^1-2^1=5$. For the induction step, suppose that P_n is true. To verify P_{n+1} we write

$$\begin{aligned}7^{n+1}-2^{n+1}&=7^{n+1}-7\cdot 2^n+7\cdot 2^n-2\cdot 2^n\\&=7\left[7^n-2^n\right]+5\cdot 2^n.\end{aligned}$$

Since 7^n-2^n is a multiple of 5 by the induction hypothesis, it follows that $7^{n+1}-2^{n+1}$ is also a multiple of 5. In fact, if $7^n-2^n=5m$, then $7^{n+1}-2^{n+1}=5[7m+2^n]$. We have shown that P_n implies P_{n+1} and so the induction step holds. An application of mathematical induction completes the proof. □

EXAMPLE 3. Show that $|\sin nx|\le n|\sin x|$ for all natural numbers n and all real numbers x.

SOLUTION. Our nth proposition is

$$P_n:\ \text{“}|\sin nx|\le n|\sin x| \text{ for all real numbers } x.\text{”}$$

The basis for induction is again clear. Suppose P_n is true. We apply the addition formula for sine to obtain

$$|\sin(n+1)x|=|\sin(nx+x)|=|\sin nx\cos x+\cos nx\sin x|.$$

Now we apply the Triangle Inequality and properties of the absolute value [see 3.7 and 3.5] to obtain

$$|\sin(n+1)x| \leqslant |\sin nx|\cdot|\cos x| + |\cos nx|\cdot|\sin x|.$$

Since $|\cos y| \leqslant 1$ for all y we see that

$$|\sin(n+1)x| \leqslant |\sin nx| + |\sin x|.$$

Now we apply the induction hypothesis P_n to obtain

$$|\sin(n+1)x| \leqslant n|\sin x| + |\sin x| = (n+1)|\sin x|.$$

Thus P_{n+1} holds. Finally, the result holds for all n by mathematical induction. □

Exercises

1.1. Prove $1^2+2^2+\cdots+n^2=n(n+1)(2n+1)/6$ for all natural numbers n.

1.2. Prove $3+11+\cdots+(8n-5)=4n^2-n$ for all natural numbers n.

1.3. Prove $1^3+2^3+\cdots+n^3=(1+2+\cdots+n)^2$ for all natural numbers n.

1.4. (a) Guess a formula for $1+3+\cdots+(2n-1)$ by evaluating the sum for $n=1$, 2, 3, and 4. [For $n=1$, the sum is simply 1.]
(b) Prove your formula using mathematical induction.

1.5. Prove $1+1/2+1/4+\cdots+1/2^n=2-1/2^n$ for all natural numbers n.

1.6. Prove that $(11)^n-4^n$ is divisible by 7 when n is a natural number.

1.7. Prove that 7^n-6n-1 is divisible by 36 for all positive integers n.

1.8. The principle of mathematical induction can be extended as follows. A list $P_m, P_{m+1}, \ldots$ of propositions is true provided (i) P_m is true, (ii) P_{n+1} is true whenever P_n is true and $n \geqslant m$.
(a) Prove that $n^2 > n+1$ for all integers $n \geqslant 2$.
(b) Prove that $n! > n^2$ for all integers $n \geqslant 4$. [Recall $n! = n(n-1)\cdots 3\cdot 2\cdot 1$; for example, $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$.]

1.9. (a) Decide for which integers the inequality $2^n > n^2$ is true.
(b) Prove your claim in (a) by mathematical induction.

1.10. Prove $(2n+1)+(2n+3)+(2n+5)+\cdots+(4n-1)=3n^2$ for all positive integers n.

1.11. For each $n\in\mathbb{N}$, let P_n denote the assertion "n^2+5n+1 is an even integer."
(a) Prove that P_{n+1} is true whenever P_n is true.
(b) For which n is P_n actually true? What is the moral of this exercise?

1.12. For $n\in\mathbb{N}$, let $n!$ [read "n factorial"] denote the product $1\cdot 2\cdot 3\cdots n$. Also let $0! = 1$ and define

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad \text{for } k=0,1,\ldots,n.$$

The *binomial theorem* asserts that

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$
$$= a^n + na^{n-1}b + \tfrac{1}{2}n(n-1)a^{n-2}b^2 + \cdots + nab^{n-1} + b^n.$$

(a) Verify the binomial theorem for $n = 1, 2,$ and 3.
(b) Show that $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ for $k = 1, 2, \ldots, n$.
(c) Prove the binomial theorem using mathematical induction and part (b).

§2. The Set $\mathbb{Q}$ of Rational Numbers

Small children first learn to add and to multiply natural numbers. After subtraction is introduced, the need to expand the number system to include 0 and negative integers becomes apparent. At this point the world of numbers is enlarged to include the set $\mathbb{Z}$ of all *integers*. Thus we have $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$.

Soon the space $\mathbb{Z}$ also becomes inadequate when division is introduced. The solution is to enlarge the world of numbers to include all fractions. Accordingly, we study the space $\mathbb{Q}$ of all *rational numbers*, i.e., numbers of the form m/n where $m, n \in \mathbb{Z}$ and $n \neq 0$. Note that $\mathbb{Q}$ contains all terminating decimals such as $1.492 = 1492/1000$. The connection between decimals and real numbers is discussed in 10.3 and §16. The space $\mathbb{Q}$ is a highly satisfactory algebraic system in which the basic operations addition, multiplication, subtraction and division can be fully studied. No system is perfect, however, and $\mathbb{Q}$ is inadequate in some ways. In this section we will consider the defects of $\mathbb{Q}$. In the next section we will stress the good features of $\mathbb{Q}$ and then move on to the system of real numbers.

The set $\mathbb{Q}$ of rational numbers is a very nice algebraic system until one tries to solve equations like $x^2 = 2$. It turns out that no rational number satisfies this equation and yet there are good reasons to believe that some kind of number satisfies this equation. Consider, for example, a square with sides having length one; see Figure 2.1. If d represents the length of the diagonal, then from geometry we know that $1^2 + 1^2 = d^2$, i.e., $d^2 = 2$.

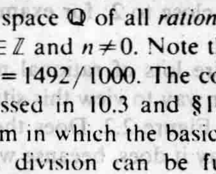

Figure 2.1

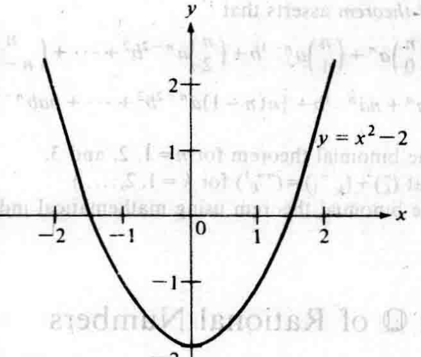

Figure 2.2

Apparently there is a positive length whose square is 2, which we write as $\sqrt{2}$. But $\sqrt{2}$ cannot be a rational number, as we will show in Example 2. Of course, $\sqrt{2}$ can be approximated by rational numbers. There are rational numbers whose squares are close to 2; for example, $(1.4142)^2 = 1.99996164$ and $(1.4143)^2 = 2.00024449$.

It is evident that there are lots of rational numbers and yet there are "gaps" in $\mathbb{Q}$. Here is another way to view this situation. Consider the graph of the polynomial x^2-2 in Figure 2.2. Does the graph of x^2-2 cross the x-axis? We are inclined to say it does, because when we draw the x-axis we include "all" the points. We allow no "gaps." But notice that the graph of x^2-2 slips by all the rational numbers on the x-axis. The x-axis is our picture of the number line and the set of rational numbers again appears to have significant "gaps."

There are even more exotic numbers such as π and e that are not rational numbers, but which come up naturally in mathematics. The number π is basic to the study of circles and spheres and e arises in problems of exponential growth.

We return to $\sqrt{2}$. This is an example of what is called an algebraic number because it satisfies the equation $x^2-2=0$.

2.1 Definition. A number is called an *algebraic number* if it satisfies a polynomial equation

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$$

where the coefficients $a_0, a_1, \ldots, a_n$ are integers, $a_n \neq 0$ and $n \geqslant 1$.

Rational numbers are always algebraic numbers. In fact, if $r = m/n$ is a rational number [$m, n \in \mathbb{Z}$ and $n \neq 0$], then it satisfies the equation $nx - m$

= 0. Numbers defined in terms of $\sqrt{\ }$, $\sqrt[3]{\ }$, etc. [or fractional exponents, if you prefer] and ordinary algebraic operations on the rational numbers are invariably algebraic numbers.

EXAMPLE 1. $4/17$, $3^{1/2}$, $(17)^{1/3}$, $(2+5^{1/3})^{1/2}$ and $((4-2\cdot 3^{1/2})/7)^{1/2}$ all represent algebraic numbers. In fact, $4/17$ is a solution of $17x-4=0$, $3^{1/2}$ represents a solution of $x^2-3=0$, and $(17)^{1/3}$ represents a solution of $x^3-17=0$. The expression $a=(2+5^{1/3})^{1/2}$ means $a^2=2+5^{1/3}$ or $a^2-2=5^{1/3}$ so that $(a^2-2)^3=5$. Therefore we have $a^6-6a^4+12a^2-13=0$ which shows that $a=(2+5^{1/3})^{1/2}$ satisfies the polynomial equation $x^6-6x^4+12x^2-13=0$. Similarly, the expression $b=((4-2\cdot 3^{1/2})/7)^{1/2}$ leads to $7b^2=4-2\cdot 3^{1/2}$, hence $2\cdot 3^{1/2}=4-7b^2$, hence $12=(4-7b^2)^2$, hence $49b^4-56b^2+4=0$. Thus b satisfies the polynomial equation $49x^4-56x^2+4=0$.

The next theorem may be familiar from elementary algebra. It is the theorem that justifies the following remarks: the only possible rational solutions of $x^3-7x^2+2x-12=0$ are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$ and so the only possible (rational) monomial factors of $x^3-7x^2+2x-12$ are $x-1, x+1, x-2, x+2, x-3, x+3, x-4, x+4, x-6, x+6, x-12, x+12$. We won't pursue these algebraic problems; we merely made these observations in the hope that they would be familiar.

The next theorem also allows one to prove that algebraic numbers that do not look like rational numbers are not rational numbers. Thus $\sqrt{4}$ is obviously a rational number, while $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc. turn out to be nonrational. See the examples following the theorem. Recall that an integer k is a *factor* of an integer m or *divides* m if m/k is also an integer. An integer $p \geqslant 2$ is a *prime* provided the only positive factors of p are 1 and p. It can be shown that every positive integer can be written as a product of primes and that this can be done in only one way.

2.2 Rational Zeros Theorem. *Suppose that $a_0, a_1, \ldots, a_n$ are integers and that r is a rational number satisfying the polynomial equation*

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{1}$$

where $a_n \neq 0$ and $n \geqslant 1$. Write $r=p/q$ where p, q are integers having no common factors and $q \neq 0$. Then q divides a_n and p divides a_0.

In other words, the only rational *candidates* for solutions of (1) have the form p/q where p divides a_0 and q divides a_n.

PROOF. We are given

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0.$$

We multiply through by q^n and obtain

$$a_n p^n + a_{n-1} p^{n-1} q + a_{n-2} p^{n-2} q^2 + \cdots + a_2 p^2 q^{n-2} + a_1 p q^{n-1} + a_0 q^n = 0. \tag{2}$$

If we solve for $a_n p^n$, we obtain

$$a_n p^n = -q\left[a_{n-1} p^{n-1} + a_{n-2} p^{n-2} q + \cdots + a_2 p^2 q^{n-3} + a_1 p q^{n-2} + a_0 q^{n-1} \right].$$

It follows that q divides $a_n p^n$. But p and q have no common factors and so q must divide a_n. [Here are more details: p can be written as a product of primes $p_1 p_2 \cdots p_k$ where the p_i's need not be distinct. Likewise q can be written as a product of primes $q_1 q_2 \cdots q_l$. Since q divides $a_n p^n$, the quantity $a_n p^n / q = a_n p_1^n \cdots p_k^n / (q_1 \cdots q_l)$ must be an integer. Since no p_i can equal any q_j, the unique factorization of a_n as a product of primes must include the product $q_1 q_2 \cdots q_l$. Thus q divides a_n.]

Now we solve (2) for $a_0 q^n$ and obtain

$$a_0 q^n = -p\left[a_n p^{n-1} + a_{n-1} p^{n-2} q + \cdots + a_2 p q^{n-2} + a_1 q^{n-1} \right].$$

Thus p divides $a_0 q^n$. Since p and q have no common factors, p must divide a_0. □

EXAMPLE 2. $\sqrt{2}$ cannot represent a rational number.

PROOF. By Theorem 2.2 the only rational numbers that could possibly be solutions of $x^2 - 2 = 0$ are ± 1, ± 2. [Here $n = 2$, $a_2 = 1$, $a_1 = 0$, $a_0 = -2$. So rational solutions must have the form p/q where p divides $a_0 = -2$ and q divides $a_2 = 1$.] One can substitute each of the four numbers ± 1, ± 2 into the equation $x^2 - 2 = 0$ to quickly eliminate them as possible solutions of this equation. Since $\sqrt{2}$ represents a solution of $x^2 - 2 = 0$, it cannot represent a rational number. □

EXAMPLE 3. $\sqrt{17}$ cannot represent a rational number.

PROOF. The only possible rational solutions of $x^2 - 17 = 0$ are ± 1, ± 17 and none of these numbers are solutions. □

EXAMPLE 4. $6^{1/3}$ cannot represent a rational number.

PROOF. The only possible rational solutions of $x^3 - 6 = 0$ are ± 1, ± 2, ± 3, ± 6. It is easy to verify that none of these eight numbers satisfies the equation $x^3 - 6 = 0$. □

EXAMPLE 5. $a = (2 + 5^{1/3})^{1/2}$ does not represent a rational number.

PROOF. In Example 1 we showed that a represents a solution of $x^6 - 6x^4 + 12x^2 - 13 = 0$. By Theorem 2.2, the only possible rational solutions are ± 1, ± 13. When $x = 1$ or -1, the left hand side of the equation is -6 and when $x = 13$ or -13, the left hand side of the equation turns out to equal 4,657,458. This last computation could be avoided by using a little common sense. Either observe that a is "obviously" bigger than 1 and less than 13, or observe that

$$13^6 - 6 \cdot 13^4 + 12 \cdot 13^2 - 13 = 13(13^5 - 6 \cdot 13^3 + 12 \cdot 13 - 1) \neq 0$$

since the term in parentheses cannot be zero: it is one less than some multiple of 13. □

EXAMPLE 6. $b = ((4 - 2\sqrt{3})/7)^{1/2}$ does not represent a rational number.

PROOF. In Example 1 we showed that b is a solution of $49x^4 - 56x^2 + 4 = 0$. The possible rational solutions of this equation are ± 1, $\pm 1/7$, $\pm 1/49$, ± 2, $\pm 2/7$, $\pm 2/49$, ± 4, $\pm 4/7$, $\pm 4/49$. To complete our proof all we need to do is substitute these eighteen candidates into the equation $49x^4 - 56x^2 + 4 = 0$. This prospect is so discouraging, however, that we choose to find a more clever approach. In Example 1, we also showed that $12 = (4 - 7b^2)^2$. Now if b were rational, then $4 - 7b^2$ would also be rational [Exercise 2.6] and so the equation $12 = x^2$ would have a rational solution. But the only possible rational solutions to $x^2 - 12 = 0$ are ± 1, ± 2, ± 3, ± 4, ± 6, ± 12 and these can all be eliminated by mentally substituting them into the equation. We conclude that $4 - 7b^2$ cannot be rational and so b cannot be rational. □

As a practical matter, many or all of the rational candidates given by the Rational Zeros Theorem can be eliminated by approximating the quantity in question [perhaps with the aid of a calculator]. It is nearly obvious that the values in Examples 2 through 5 are not integers while all the rational candidates are. My calculator says that b in Example 6 is approximately .2767; the nearest rational candidate is $+2/7$ which is approximately .2857.

EXERCISES

2.1. Show that $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{24}$, and $\sqrt{31}$ are not rational numbers.

2.2. Show that $2^{1/3}$, $5^{1/7}$, and $(13)^{1/4}$ do not represent rational numbers.

2.3. Show that $(2 + \sqrt{2})^{1/2}$ does not represent a rational number.

2.4. Show that $(5 - \sqrt{3})^{1/3}$ does not represent a rational number.

2.5. Show that $[3 + \sqrt{2}]^{2/3}$ does not represent a rational number.

2.6. In connection with Example 6, discuss why $4 - 7b^2$ must be rational if b is rational.

§3. The Set $\mathbb{R}$ of Real Numbers

The set $\mathbf{Q}$ is probably the largest system of numbers with which you really feel comfortable. There are subtleties but you have learned to cope with them. For example, $\mathbf{Q}$ is not simply the *set* $\{m/n : m, n \in \mathbb{Z}, n \neq 0\}$ since we regard some pairs of different looking fractions as equal. For example, $2/4$ and $3/6$ are regarded as the same element of $\mathbf{Q}$. A rigorous development of $\mathbf{Q}$ based on $\mathbb{Z}$, which in turn is based on $\mathbb{N}$, would require us to introduce the notion of equivalence class; see [19]. In this book we assume a familiarity with and understanding of $\mathbf{Q}$ as an algebraic system. However, in order to clarify exactly what we need to know about $\mathbf{Q}$, we set down some of its basic axioms and properties.

The basic algebraic operations in $\mathbf{Q}$ are addition and multiplication. Given a pair a, b of rational numbers, the sum $a+b$ and the product ab also represent rational numbers. Moreover, the following properties hold.

A1. $a+(b+c)=(a+b)+c$ for all a, b, c.
A2. $a+b=b+a$ for all a, b.
A3. $a+0=a$ for all a.
A4. For each a, there is an element $-a$ such that $a+(-a)=0$.
M1. $a(bc)=(ab)c$ for all a, b, c.
M2. $ab=ba$ for all a, b.
M3. $a \cdot 1=a$ for all a.
M4. For each $a \neq 0$, there is an element a^{-1} such that $aa^{-1}=1$.
DL. $a(b+c)=ab+ac$ for all a, b, c.

Properties A1 and M1 are called the *associative laws* and properties A2 and M2 are the *commutative laws*. Property DL is the *distributive law*; this is the least obvious law and is the one that justifies "factorization" and "multiplying out" in algebra. A system that has more than one element and satisfies these nine properties is called a *field*. The basic algebraic properties of $\mathbf{Q}$ can be proved solely on the basis of these field properties. We do not want to pursue this topic in any depth, but we illustrate our claim by proving some familiar properties in Theorem 3.1 below.

The set $\mathbf{Q}$ also has an order structure $\leq$ satisfying

O1. Given a and b, either $a \leq b$ or $b \leq a$.
O2. If $a \leq b$ and $b \leq a$, then $a=b$.
O3. If $a \leq b$ and $b \leq c$, then $a \leq c$.
O4. If $a \leq b$, then $a+c \leq b+c$.
O5. If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

Property O3 is called the *transitive law*. This is the characteristic property of an ordering. A field with an ordering satisfying properties O1 through O5 is called an *ordered field*. Most of the algebraic and order properties of $\mathbf{Q}$ can be established for any ordered field. We will prove a few of them in Theorem 3.2 below.