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SOBOLEV SPACES

SECOND EDITION

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Second Edition

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SOBOLEV SPACES

Second Edition

To Anne and Frances

who had to put up with it all

This is volume 140 in the PURE AND APPLIED MATHEMATICS series
Founding Editors: Paul A. Smith and Samuel Eilenberg

PREFACE

This monograph presents an introductory study of of the properties of certain Banach spaces of weakly differentiable functions of several real variables that arise in connection with numerous problems in the theory of partial differential equations, approximation theory, and many other areas of pure and applied mathematics. These spaces have become associated with the name of the late Russian mathematician S. L. Sobolev, although their origins predate his major contributions to their development in the late 1930s.

Even by 1975 when the first edition of this monograph was published, there was a great deal of material on these spaces and their close relatives, though most of it was available only in research papers published in a wide variety of journals. The monograph was written to fill a perceived need for a single source where graduate students and researchers in a wide variety of disciplines could learn the essential features of Sobolev spaces that they needed for their particular applications. No attempt was made even at that time for complete coverage. To quote from the Preface of the first edition:

> The existing mathematical literature on Sobolev spaces and their generalizations is vast, and it would be neither easy nor particularly desirable to include everything that was known about such spaces between the covers of one book. An attempt has been made in this monograph to present all the core material in sufficient generality to cover most applications, to give the reader an overview of the subject that is difficult to obtain by reading research papers, and finally ... to provide a ready reference for someone requiring a result about Sobolev spaces for use in some application.

This remains as the purpose and focus of this second edition. During the intervening twenty-seven years the research literature has grown exponentially, and there are now several other books in English that deal in whole or in part with Sobolev spaces. (For example, see [Ad2], [Bu1], [Mz1], [Tr1], [Tr3], and [Tr4].) However, there is still a need for students in other disciplines than mathematics, and in other areas of mathematics than just analysis to have available a book that describes these spaces and their core properties based only a background in mathematical analysis at the senior undergraduate level. We have tried to make this such a book.

The organization of this book is similar but not identical to that of the first edition:

Chapter 1 remains a potpourri of standard topics from real and functional analysis, included, mainly without proofs, because they provide a necessary background for what follows.

Chapter 2 on the Lebesgue Spaces $L^p(\Omega)$ is much expanded and reworked from the previous edition. It provides, in addition to standard results about these spaces, a brief treatment of mixed-norm L^p spaces, weak-L^p spaces, and the Marcinkiewicz interpolation theorem, all of which will be used in a new treatment of the Sobolev Imbedding Theorem in Chapter 4. For the most part, complete proofs are given, as they are for much of the rest of the book.

Chapter 3 provides the basic definitions and properties of the Sobolev spaces $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$. There are minor changes from the first edition.

Chapter 4 is now completely concerned with the imbedding properties of Sobolev Spaces. The first half gives a more streamlined presentation and proof of the various imbeddings of Sobolev spaces into L^p spaces, including traces on subspaces of lower dimension, and spaces of continuous and uniformly continuous functions. Because the approach to the Sobolev Imbedding Theorem has changed, the roles of Chapters 4 and 5 have switched from the first edition. The latter part of Chapter 4 deals with situations where the regularity conditions on the domain Ω that are necessary for the full Sobolev Imbedding Theorem do not apply, but some weaker imbedding results are still possible.

Chapter 5 now deals with interpolation, extension, and approximation results for Sobolev spaces. Part of it is expanded from material in Chapter 4 of the first edition with newer results and methods of proof.

Chapter 6 deals with establishing compactness of Sobolev imbeddings. It is only slightly changed from the first edition.

Chapter 7 is concerned with defining and developing properties of scales of spaces with fractional orders of smoothness, rather than the integer orders of the Sobolev spaces themselves. It is completely rewritten and bears little resemblance to the corresponding chapter in the first edition. Much emphasis is placed on real interpolation methods. The J-method and K-method are fully presented and used to develop the theory of Lorentz spaces and Besov spaces and their imbeddings, but both families of spaces are also provided with intrinsic characterizations. A key theorem identifies lower dimensional traces of functions in Sobolev spaces as constituting certain Besov spaces. Complex interpolation is used to introduce Sobolev spaces of fractional order (also called spaces of Bessel potentials) and Fourier transform methods are used to characterize and generalize these spaces to yield the Triebel Lizorkin spaces and illuminate their relationship with the Besov spaces.

Chapter 8 is very similar to its first edition counterpart. It deals with Orlicz and Orlicz-Sobolev spaces which generalize L^p and $W^{m,p}$ spaces by allowing the role of the function t^p to be assumed by a more general convex function $A(t)$. An important result identifies a certain Orlicz space as a target for an imbedding of $W^{m,p}(\Omega)$ in a limiting case where there is an imbedding into $L^p(\Omega)$ for $1 \le p < \infty$ but not into $L^\infty(\Omega)$.

This monograph was typeset by the authors using TEX on a PC running Linux-Mandrake 8.2. The figures were generated using the mathematical graphics software package *MG* developed by R. B. Israel and R. A. Adams.

RAA & JJFF
Vancouver, August 2002

List of Spaces and Norms

Space	Norm	Paragraph
$B^{s;p,q}(\Omega)$	$\|\cdot\,; B^{s;p,q}(\Omega)\|$	7.32
$B^{s;p,q}(\mathbb{R}^n)$	$\|\cdot\,; B^{s;p,q}(\mathbb{R}^n)\|$	7.67
$\dot{B}^{s;p,q}(\mathbb{R}^n)$		7.68
$C^m(\Omega),\ C^\infty(\Omega)$		1.26
$C_0(\Omega),\ C_0^\infty(\Omega)$		1.26
$C^m(\overline{\Omega})$	$\|\cdot\,; C^m(\overline{\Omega})\|$	1.28
$C^{m,\lambda}(\overline{\Omega})$	$\|\cdot\,; C^{m,\lambda}(\overline{\Omega})\|$	1.29
$C_B^m(\Omega)$	$\|\cdot\,; C_B^m(\Omega)\|$	1.27, 4.2
$C^j(\overline{\Omega})$	$\|\cdot\,; C^j(\overline{\Omega})\|$	4.2
$C^{j,\lambda}(\overline{\Omega})$	$\|\cdot\,; C^{j,\lambda}(\overline{\Omega})\|$	4.2
$C^{j,\lambda,q}(\overline{\Omega})$	$\|\cdot\,; C^{j,\lambda,q}(\overline{\Omega})\|$	7.35
$\mathscr{D}(\Omega)$		1.56
$\mathscr{D}'(\Omega)$		1.57
$E_A(\Omega)$	$\|\cdot\|_A = \|\cdot\|_{A,\Omega}$	8.14

$F^{s;p,q}(\Omega)$	$\left\|\cdot\,; F^{s;p,q}(\Omega)\right\|$	7.69
$F^{s;p,q}(\mathbb{R}^n)$	$\left\|\cdot\,; F^{s;p,q}(\mathbb{R}^n)\right\|$	7.65
$\dot{F}^{s;p,q}(\mathbb{R}^n)$		7.66
$H^{m,p}(\Omega)$	$\|\cdot\|_{m,p} = \|\cdot\|_{m,p,\Omega}$	3.2
$L_A(\Omega)$	$\|\cdot\|_A = \|\cdot\|_{A,\Omega}$	8.9
$L^p(\Omega)$	$\|\cdot\|_p = \|\cdot\|_{p,\Omega}$	2.1, 2.3
$L^{\mathbf{p}}(\mathbb{R}^n)$	$\|\cdot\|_{\mathbf{p}}$	2.48
$L^\infty(\Omega)$	$\|\cdot\|_\infty = \|\cdot\|_{\infty,\Omega}$	2.10
$L^q(a, b; d\mu, X)$	$\|\cdot\,; L^q(a, b; d\mu, X)\|$	7.4
L^q_*	$\left\|\cdot\,; L^q_*\right\|$	7.5
$L^1_{\mathrm{loc}}(\Omega)$		1.58
$L^{p,q}(\Omega)$	$\|\cdot\,; L^{p,q}(\Omega)\|$	7.25
ℓ^p	$\|\cdot\,; \ell^p\|$	2.27
$\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$		7.59
weak-$L^p(\Omega)$	$[\cdot]_p = [\cdot]_{p,\Omega}$	2.55
$W^{m,p}(\Omega)$	$\|\cdot\|_{m,p} = \|\cdot\|_{m,p,\Omega}$	3.2
$W^{m,p}_0(\Omega)$	$\|\cdot\|_{m,p} = \|\cdot\|_{m,p,\Omega}$	3.2
$W^{-m,p'}(\Omega)$	$\|\cdot\|_{-m,p'}$	3.12, 3.13
$W^m E_A(\Omega)$	$\|\cdot\|_{m,A} = \|\cdot\|_{m,A,\Omega}$	8.30
$W^m L_A(\Omega)$	$\|\cdot\|_{m,A} = \|\cdot\|_{m,A,\Omega}$	8.30
$W^{s,p}(\Omega)$	$\|\cdot\,; W^{s,p}(\Omega)\|$	7.57
$W^{s,p}(\mathbb{R}^n)$	$\|\cdot\,; W^{s,p}(\mathbb{R}^n)\|$	7.64
X	$\|\cdot\,; X\|$	1.7
$X_0 \cap X_1$	$\|\cdot\|_{X_0 \cap X_1}$	7.7
$X_0 + X_1$	$\|\cdot\|_{X_0 + X_1}$	7.7
$(X_0, X_1)_{\theta,q;J}$	$\|\cdot\|_{\theta,q;J}$	7.13
$(X_0, X_1)_{\theta,q;K}$	$\|\cdot\|_{\theta,q;K}$	7.10
$[X_0, X_1]_\theta$	$\|u\|_{[X_0,X_1]_\theta}$	7.51
$X_0^{1-\theta} X_1^\theta$	$\left\|\cdot\,; X_0^{1-\theta} X_1^\theta\right\|$	7.54

CONTENTS

1

PRELIMINARIES

1.1 (Introduction) Sobolev spaces are vector spaces whose elements are functions defined on domains in n-dimensional Euclidean space $\mathbb{R}^n$ and whose partial derivatives satisfy certain integrability conditions. In order to develop and elucidate the properties of these spaces and mappings between them we require some of the machinery of general topology and real and functional analysis. We assume that readers are familiar with the concept of a vector space over the real or complex scalar field, and with the related notions of dimension, subspace, linear transformation, and convex set. We also expect the reader will have some familiarity with the concept of topology on a set, at least to the extent of understanding the concepts of an open set and continuity of a function.

In this chapter we outline, mainly without any proofs, those aspects of the theories of topological vector spaces, continuity, the Lebesgue measure and integral, and Schwartz distributions that will be needed in the rest of the book. For a reader familiar with the basics of these subjects, a superficial reading to settle notations and review the main results will likely suffice.

Notation

1.2 Throughout this monograph the term *domain* and the symbol Ω will be reserved for a nonempty open set in n-dimensional real Euclidean space $\mathbb{R}^n$. We shall be concerned with the differentiability and integrability of functions defined on Ω; these functions are allowed to be complex-valued unless the contrary is explicitly stated. The complex field is denoted by $\mathbb{C}$. For $c \in \mathbb{C}$ and two functions u and v, the scalar multiple cu, the sum $u + v$, and the product uv are always defined pointwise:

$$(cu)(x) = cu(x),$$
$$(u + v)(x) = u(x) + v(x),$$
$$(uv)(x) = u(x)v(x)$$

at all points x where the right sides make sense.

A typical point in $\mathbb{R}^n$ is denoted by $x = (x_1, \ldots, x_n)$; its norm is given by $|x| = (\sum_{j=1}^n x_j^2)^{1/2}$. The inner product of two points x and y in $\mathbb{R}^n$ is $x \cdot y = \sum_{j=1}^n x_j y_j$.

If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers α_j, we call α a *multi-index* and denote by x^α the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, which has degree $|\alpha| = \sum_{j=1}^n \alpha_j$. Similarly, if $D_j = \partial/\partial x_j$, then

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$$

denotes a differential operator of order $|\alpha|$. Note that $D^{(0,\ldots,0)}u = u$.

If α and β are two multi-indices, we say that $\beta \le \alpha$ provided $\beta_j \le \alpha_j$ for $1 \le j \le n$. In this case $\alpha - \beta$ is also a multi-index, and $|\alpha - \beta| + |\beta| = |\alpha|$. We also denote

$$\alpha! = \alpha_1! \cdots \alpha_n!$$

and if $\beta \le \alpha$,

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}.$$

The reader may wish to verify the Leibniz formula

$$D^\alpha(uv)(x) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta u(x) D^{\alpha-\beta} v(x)$$

valid for functions u and v that are $|\alpha|$ times continuously differentiable near x.

1.3 If $G \subset \mathbb{R}^n$ is nonempty, we denote by $\overline{G}$ the closure of G in $\mathbb{R}^n$. We shall write $G \Subset \Omega$ if $\overline{G} \subset \Omega$ and $\overline{G}$ is a compact (that is, closed and bounded) subset of $\mathbb{R}^n$. If u is a function defined on G, we define the *support* of u to be the set

$$\operatorname{supp}(u) = \overline{\{x \in G : u(x) \neq 0\}}.$$

We say that u has *compact support* in Ω if $\operatorname{supp}(u) \Subset \Omega$. We denote by "bdry G" the boundary of G in $\mathbb{R}^n$, that is, the set $\overline{G} \cap \overline{G^c}$, where G^c is the complement of G in $\mathbb{R}^n$; $G^c = \mathbb{R}^n - G = \{x \in \mathbb{R}^n : x \notin G\}$.

If $x \in \mathbb{R}^n$ and $G \subset \mathbb{R}^n$, we denote by "dist(x, G)" the distance from x to G, that is, the number $\inf_{y \in G} |x - y|$. Similarly, if $F, G \subset \mathbb{R}^n$ are both nonempty,

$$\operatorname{dist}(F, G) = \inf_{y \in F} \operatorname{dist}(y, G) = \inf_{\substack{x \in G \\ y \in F}} |y - x|.$$

Topological Vector Spaces

1.4 (Topological Spaces) If X is any set, a *topology* on X is a collection $\mathcal{O}$ of subsets of X which contains

(i) the whole set X and the empty set $\emptyset$,

(ii) the union of any collection of its elements, and

(iii) the intersection of any finite collection of its elements.

The pair $(X, \mathcal{O})$ is called a *topological space* and the elements of $\mathcal{O}$ are the *open sets* of that space. An open set containing a point x in X is called a *neighbourhood* of x. The complement $X - U = \{x \in X : x \notin U\}$ of any open set U is called a *closed* set. The closure $\overline{S}$ of any subset $S \subset X$ is the smallest closed subset of X that contains S.

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two topologies on the same set X. If $\mathcal{O}_1 \subset \mathcal{O}_2$, we say that $\mathcal{O}_2$ is *stronger* than $\mathcal{O}_1$, or that $\mathcal{O}_1$ is *weaker* than $\mathcal{O}_2$.

A topological space $(X, \mathcal{O})$ is called a *Hausdorff space* if every pair of distinct points x and y in X have disjoint neighbourhoods.

The *topological product* of two topological spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is the topological space $(X \times Y, \mathcal{O})$, where $X \times Y = \{(x, y) : x \in X, \ y \in Y\}$ is the Cartesian product of the sets X and Y, and $\mathcal{O}$ consists of arbitrary unions of sets of the form $\{O_X \times O_Y : O_X \in \mathcal{O}_X, \ O_Y \in \mathcal{O}_Y\}$.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two topological spaces. A function f from X into Y is said to be *continuous* if the preimage $f^{-1}(O) = \{x \in X : f(x) \in O\}$ belongs to $\mathcal{O}_X$ for every $O \in \mathcal{O}_Y$. Evidently the stronger the topology on X or the weaker the topology on Y, the more such continuous functions f there will be.

1.5 (Topological Vector Spaces) We assume throughout this monograph that all vectors spaces referred to are taken over the complex field unless the contrary is explicitly stated.

A *topological vector space*, hereafter abbreviated TVS, is a Hausdorff topological space that is also a vector space for which the vector space operations of addition and scalar multiplication are continuous. That is, if X is a TVS, then the mappings

$$(x, y) \to x + y \qquad \text{and} \qquad (c, x) \to cx$$

from the topological product spaces $X \times X$ and $\mathbb{C} \times X$, respectively, into X are continuous. (Here $\mathbb{C}$ has its usual topology induced by the Euclidean metric.)

X is a *locally convex* TVS if each neighbourhood of the origin in X contains a convex neighbourhood of the origin.

We outline below those aspects of the theory of topological and normed vector spaces that play a significant role in the study of Sobolev spaces. For a more thorough discussion of these topics the reader is referred to standard textbooks on functional analysis, for example [Ru1] or [Y].

1.6 (Functionals) A scalar-valued function defined on a vector space X is called a *functional*. The functional f is linear provided

$$f(ax + by) = af(x) + bf(y), \qquad x, y \in X, \quad a, b \in \mathbb{C}.$$

If X is a TVS, a functional on X is continuous if it is continuous from X into $\mathbb{C}$ where $\mathbb{C}$ has its usual topology induced by the Euclidean metric.

The set of all continuous, linear functionals on a TVS X is called the *dual* of X and is denoted by X'. Under pointwise addition and scalar multiplication X' is itself a vector space:

$$(f + g)(x) = f(x) + g(x), \quad (cf)(x) = cf(x), \qquad f, g \in X', \ x \in X, \ c \in \mathbb{C}.$$

X' will be a TVS provided a suitable topology is specified for it. One such topology is the *weak-star topology*, the weakest topology with respect to which the functional F_x, defined on X' by $F_x(f) = f(x)$ for each $f \in X'$, is continuous for each $x \in X$. This topology is used, for instance, in the space of Schwartz distributions introduced in Paragraph 1.57. The dual of a normed vector space can be given a stronger topology with respect to which it is itself a normed space. (See Paragraph 1.11.)

Normed Spaces

1.7 (Norms) A *norm* on a vector space X is a real-valued function f on X satisfying the following conditions:

(i) $f(x) \geq 0$ for all $x \in X$ and $f(x) = 0$ if and only if $x = 0$,

(ii) $f(cx) = |c| f(x)$ for every $x \in X$ and $c \in \mathbb{C}$,

(iii) $f(x + y) \leq f(x) + f(y)$ for every $x, y \in X$.

A *normed space* is a vector space X provided with a norm. The norm will be denoted $\|\cdot\,; X\|$ except where other notations are introduced.

If $r > 0$, the set

$$B_r(x) = \{y \in X : \|y - x\,; X\| < r\}$$