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Recent Advances in Algebraic Geometry

A Volume in Honor of Rob Lazarsfeld's 60th Birthday

Edited by
Christopher D. Hacon, Mircea Mustaţă
and Mihnea Popa

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Edited by

CHRISTOPHER D. HACON
University of Utah

MIRCEA MUSTAŢĂ
University of Michigan

MIHNEA POPA
Northwestern University

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Contributors

Thomas Bauer *Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Hans-Meerwein-Straße, D-35032 Marburg, Germany;* tbauer@mathematik.uni-marburg.de

Aaron Bertram *Department of Mathematics, University of Utah, Salt Lake City, UT 84112-0090, USA;* bertram@math.utah.edu

Sébastien Boucksom *CNRS - Institut de Mathématiques de Jussieu, 4 place Jussieu, 75252 Paris Cedex, France;* boucksom@math.jussieu.fr

Gregory Burnham *c/o Bridgewater Associates, 1 Glendinning plane, Westport, CT 06880, USA*

Frédéric Campana *Université de Lorraine, Institut Élie Cartan, UMR 7502 du CNRS, BP 70239, 54506 Vandœuvre-lès-Nancy Cedex, France;* frederic.campana@univ-lorraine.fr

Fabrizio Catanese *Lehrstuhl Mathematik VIII, Mathematisches Institut der Universität Bayreuth, NW II, Universitätsstr. 30, 95447 Bayreuth, Germany;* Fabrizio.Catanese@uni-bayreuth.de

Jungkai Alfred Chen *National Center for Theoretical Sciences, Taipei Office and Department of Mathematics, National Taiwan University, Taipei 106, Taiwan;* jkchen@math.ntu.edu.tw

Olivier Debarre *Département de Mathématiques et Applications, École Normale Supérieure et CNRS, 45 rue d'Ulm, 75230 Paris cedex 05, France;* olivier.debarre@ens.fr

Tommaso de Fernex *Department of Mathematics, University of Utah, Salt Lake City, UT 48112-0090, USA;* defernex@math.utah.edu

Jean-Pierre Demailly *Université de Grenoble I, Institut Fourier, UMR 5582 du CNRS, BP 74, 38402 Saint-Martin d'Hères, France;* demailly@fourier.ujf-grenoble.fr

Igor Dolgachev *Department of Mathematics, University of Michigan, 525 E. University Ave., Ann Arbor, MI 49109, USA;* idolga@umich.edu

David Eisenbud *Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720;* eisenbud@math.berkeley.edu

Daniel Erman *Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA;* derman@math.wisc.edu

Charles Favre *CNRS - Centre de Mathématiques Laurent Schwartz, École Polytechnique, 91128 Palaiseau Cedex, France;* favre@math.polytechnique.fr

Daniel Greb *Ruhr-Universität Bochum, Fakultät für Mathematik, Arbeitsgruppe Algebra/Topologie, 44780 Bochum, Germany;* daniel.greb@ruhr-uni-bochum.de

Christopher D. Hacon *Department of Mathematics, University of Utah, 155 South 1400 East, Salt Lake City, UT 48112-0090, USA;* hacon@math.utah.edu

Robin Hartshorne *Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720;* robin@math.berkeley.edu

Benjamin Howard *Center for Communications Research, Institute for Defense Analysis, 805 Bunn Drive, Princeton, NJ 08540, USA;* bjhowa3@idaccr.org

Atanas Iliev *Department of Mathematics, Seoul National University, Gwanak Campus, Bldg. 27, Seoul 151-747, Korea;* ailiev2001@yahoo.com

János Kollár *Department of Mathematics, Princeton University, Princeton, NJ 08544-1000, USA;* kollar@math.princeton.edu

Sándor J Kovács *University of Washington, Department of Mathematics, Box 354350, Seattle, WA 98195-4350, USA;* skovacs@uw.edu

Alex Küronya *Budapest University of Technology and Economics, Department of Algebra. Address: Albert-Ludwigs-Universität Freiburg, Mathematisches Institut, Eckerstraße 1, D-79104 Freiburg, Germany;* alex.kueronya@math.uni-freiburg.de

Luigi Lombardi *Mathematisches Institut, Universität Bonn, Endenicher Allee 60, Bonn 53115, Germany;* lombardi@math.uni-bonn.de

Laurent Manivel *Institut Fourier, Université de Grenoble I et CNRS, BP 74, 38402 Saint-Martin d'Hères, France;* laurent.manivel@ujf-grenoble.fr

Shigeru Mukai *Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan;* mukai@kurims.kyoto-u.ac.jp

Hisanori Ohashi *Department of Mathematics, Faculty of Science and Technology, Tokyo University of Science, 2641 Yamazaki, Noda, Chiba 278-8510, Japan;* ohashi@ma.noda.tus.ac.jp, ohashi.hisanori@gmail.com

Roberto Paoletti *Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano Bicocca, Via R. Cozzi 53, 20125 Milano, Italy;* `roberto.paoletti@unimib.it`

Giuseppe Pareschi *Dipartimento di Matematica, Università di Roma, Tor Vergata, Viale della Ricerca Scientifica, 00133 Roma, Italy;* `pareschi@axp.mat.uniroma2.it`

Thomas Peternell *Universität Bayreuth, Mathematisches Institut, D-95440 Bayreuth, Germany;* `thomas.peternell@uni-bayreuth.de`

Mihnea Popa *Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208-2730, USA;* `mpopa@math.northwestern.edu`

Zvi Rosen *Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720, USA;* `zhrosen@math.berkeley.edu`

Christian Schnell *Department of Mathematics, Stony Brook University, Stony Brook, NY 11794, USA;* `cschnell@math.sunysb.edu`

Frank-Olaf Schreyer *Mathematik und Informatik, Universität des Saarlandes, Campus E2 4, D-66123 Saarbrücken, Germany;* `schreyer@math.uni-sb.de`

Jessica Sidman *Department of Mathematics and Statistics, Mount Holyoke College, South Hadley, MA 01075, USA;* `jsidman@mtholyoke.edu`

Tomasz Szemberg *Instytut Matematyki UP, Podchorążych 2, PL-30-084 Kraków, Poland;* `tomasz.szemberg@gmail.com`

Stefano Urbinati *Università degli Studi di Padova, Via Trieste 63, 35121 Padova, Italy;* `urbinati.st@gmail.com`

Peter Vermeire *Department of Mathematics, Central Michigan University, Mount Pleasant, MI 48859, USA;* `p.vermeire@cmich.edu`

Claire Voisin *CNRS and École Polytechnique, Centre de mathématiques Laurent Schwartz, 91128 Palaiseau Cédex, France;* `voisin@math.polytechnique.fr`

Johnathan Wahl *University of North Carolina at Chapel Hill, Chapel Hill, NC 27599-3250, USA;* `jmwahl@email.unc.edu`

Preface

The conference "Recent Advances in Algebraic Geometry" was held between May 16 and 19, 2013, at the University of Michigan, Ann Arbor, to honor Robert K. Lazarsfeld (known as "Rob" among friends and colleagues) on the occasion of his 60th birthday. The conference honored Rob's outstanding contributions to algebraic geometry and the mathematical community, bringing together a large crowd, including many of his former students, collaborators, colleagues, and friends. It was a happy occasion for many of us who have known Rob and have been touched by his influence as students and peers, or simply as members of the algebraic geometry world.

From a personal point of view, we cannot even begin to discuss Rob's career without mentioning one of its most distinguished aspects, namely the unique influence he has had on the younger generations through teaching and mentoring. His style as a doctoral advisor and as an expositor is famous throughout the algebraic geometry community. He has been the advisor of more than 20 students, has numerous other mathematical descendants, and has mentored successful postdoctoral fellows. Many of these are now established mathematicians helping to expand the boundaries of Rob's mathematical vision. His generosity and ability to generate good problems, and his active support of the careers of his students, have been for many of us some of the most crucial aspects of our mathematical lives.

We highlight a few reference points in Rob's mathematical career. He received his B.A. from Harvard in 1975, and his Ph.D. from Brown in 1980, under the direction of William Fulton. He then went back to Harvard as a Benjamin Peirce Assistant Professor until 1983. During the 1981–82 academic year, Rob was awarded a postdoctoral fellowship from the American Mathematical Society, which he used to visit the Institute for Advanced Study in Princeton while on leave from Harvard. There he met Lawrence Ein, with whom he would later develop a long-lasting collaboration, resulting in over 25 joint papers. In 1983, Rob moved as an Assistant Professor to UCLA, where he became a Professor in 1987. He remained at UCLA until 1997, when he joined University of Michigan. There he was named Raymond L. Wilder Collegiate Professor of Mathematics in 2007. Starting in the Fall of 2013, Rob retired from Michigan and became a Professor at Stony Brook University. Over the years, Rob has received several honors, including a Sloan Fellowship (1984–87), the National Science Foundation Young Investigator Award (1985–90), and a Guggenheim Fellowship (1998–99); he was elected to the American Academy of Arts and Sciences in 2006.

While this is not the place to give a detailed account of Rob's work and accomplishments, it is inspiring to look back and give a brief overview of some of the highlights of his research that have had a profound impact on the field. His thesis was devoted to the study of low-degree ramified coverings of projective space. At the beginning of the 1980s, in joint work with William Fulton, Rob studied positivity properties of vector bundles, with applications to classical geometric questions, such as the connectedness of Brill–Noether loci. One of the fundamental results they proved describes the positive polynomials in the Chern classes of all ample vector bundles. Around the same time Rob began his work on the Castelnuovo–Mumford regularity of smooth projective varieties. Some landmark results in this direction that he obtained over the next decade concern sharp bounds for the regularity of curves (with Gruson and Peskine, generalizing a classical result of Castelnuovo) and surfaces, as well as a sharp bound in terms of the degrees of defining equations (with Bertram and Ein). In the mid-1980s, Rob began a collaboration with Mark Green that resulted in an extraordinarily influential series of papers, generating a large amount of research in algebraic geometry to this day. Some of these papers were devoted to the study of syzygies of smooth curves embedded in projective space. They contained important results and further conjectures on the precise connection between the algebraic invariants in the form of syzygies of the embedding, and the intrinsic geometry of the curve. Others were devoted to the study of cohomological support loci for topologically trivial line bundles, proving in particular their famous generic vanishing theorem. This led to a flurry of subsequent activity, involving both extensions and a wide array of applications, ranging from the study of singularities of theta divisors to that of the birational geometry of varieties with nontrivial holomorphic one-forms.

The most significant part of Rob's work since the beginning of the 1990s, largely done jointly with Lawrence Ein, but involving numerous other collaborators as well, revolved around geometric applications of vanishing theorems. Among the many fundamental results he obtained in this area, we mention only the proof of Fujita's conjecture for threefolds, an effective geometric version of Hilbert's Nullstellensatz, the fact that theta divisors have rational singularities, as well as various applications of asymptotic multiplier ideals to effective bounds in commutative algebra. Several new concepts, phenomena, or points of view in this circle of ideas, such as the notion of a graded sequence of ideals or the asymptotic study of linear series via the volume function, have their origin in Rob's work. Over the past few years, Rob has continued to ask fundamental questions and open new avenues of exploration, especially while studying Okounkov bodies, or the asymptotic behavior of syzygies. All of us influenced by Rob's work over the years are looking forward with excitement to Rob's future results and insights.

Rob's deep influence on the field of algebraic geometry and on how we think is not solely the outcome of his research papers and his teaching. When his book *Positivity in Algebraic Geometry* was published in 2004, it became an instant classic. It succeeded wonderfully in putting together under the same heading most of the areas of classical and modern complex algebraic geometry dedicated to, or influenced by, the study of positivity. It also developed for the first time the theory of multiplier ideals in textbook form, and introduced the theory of asymptotic multiplier ideals, tools that have since become of utmost importance in birational geometry. It is universally acknowledged that this will be one of a handful of fundamental references in the field of complex algebraic geometry for decades to come.

Before concluding, we would like to acknowledge the help we have received with funding and organizing the conference. We thank the National Science Foundation for support in the form of grant DMS-1262798 and the University of Michigan for financial and logistical assistance.

The papers collected in this volume are contributions from some of Rob's closest collaborators, students, and postdocs, as well as from some of the most prominent names in the subject. The reader will recognize in these contributions the extraordinary breadth of Rob's interests and influence. On behalf of the authors, all of those present at the conference, and the algebraic geometry community in general, we dedicate this volume to Rob with warmth and gratitude!

Contents

1

The effect of points fattening in dimension three

Th. Bauer[a]
Philipps-Universität Marburg

T. Szemberg[b]
Instytut Matematyki UP

Abstract

There has recently been increased interest in understanding the relationship between the symbolic powers of an ideal and the geometric properties of the corresponding variety. While a number of results are available for the two-dimensional case, higher dimensions are largely unexplored. In the present paper we study a natural conjecture arising from a result by Bocci and Chiantini. As a first step toward understanding the higher-dimensional picture, we show that this conjecture is true in dimension three. Also, we provide examples showing that the hypotheses of the conjecture may not be weakened.

Dedicated to Robert Lazarsfeld on the occasion of his sixtieth birthday

1 Introduction

The study of the effect of points fattening was initiated by Bocci and Chiantini [3]. Roughly speaking, they considered the radical ideal I of a finite set Z of points in the projective plane, its second symbolic power $I^{(2)}$, and deduced from the comparison of algebraic invariants of these two ideals various geometric properties of the set Z. Along these lines, Dumnicki *et al.* [7] studied higher symbolic powers of I. Similar problems were studied in [1] in the bi-homogeneous setting of ideals defining finite sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

It is a natural task to try to generalize the result of Bocci and Chiantini [3, Theorem 1.1] to the higher-dimensional setting. Denoting by $\alpha(I)$ the *initial*

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degree of a homogeneous ideal I, i.e., the least degree k such that $(I)_k \neq 0$, a natural generalization reads as follows:

Conjecture 1.1 *Let Z be a finite set of points in projective space $\mathbb{P}^n$ and let I be the radical ideal defining Z. If*

$$d := \alpha(I^{(n)}) = \alpha(I) + n - 1\,, \tag{1}$$

then either

$\alpha(I) = 1$, *i.e., Z is contained in a single hyperplane H in $\mathbb{P}^n$*

or

Z consists of all intersection points (i.e., points where n hyperplanes meet) of a general configuration of d hyperplanes in $\mathbb{P}^n$, i.e., Z is a star configuration. *For any polynomial in $I^{(n)}$ of degree d, the corresponding hypersurface decomposes into d such hyperplanes.*

The term *general configuration* in the conjecture means simply that no more than n hyperplanes meet in one point. This is equivalent to the *general linear position* for points in the dual projective space corresponding to the hyperplanes in the configuration. The result of Bocci and Chiantini is the case $n = 2$ of this conjecture. As a first step toward understanding the higher-dimensional picture, we show in the present paper:

Theorem 1.2 *The conjecture is true for $n = 3$.*

The assumption on the ideal I in the theorem amounts to the two equalities

$$\alpha(I^{(2)}) = \alpha(I) + 1$$
$$\alpha(I^{(3)}) = \alpha(I^{(2)}) + 1$$

and one might be tempted to relax the assumptions to only one of them. In Section 6 we provide examples showing, however, that neither is sufficient by itself to reach the conclusion of the theorem.

Star configurations are interesting objects of study in their own right. They are defined in [10] as unions of linear subspaces of fixed codimension c in projective space $\mathbb{P}^n$ that result as subspaces where exactly c of a fixed finite set of general hyperplanes in $\mathbb{P}^n$ intersect. The case described in Conjecture 1.1 corresponds thus to the $c = n$ situation. It is natural to wonder if the following further generalization of Conjecture 1.1 might be true: If Z is a finite collection of linear subspaces of codimension $c \leqslant n$ in $\mathbb{P}^n$ with the radical ideal I and such that

$$d = \alpha(I^{(c)}) = \alpha(I) + c - 1\,,$$