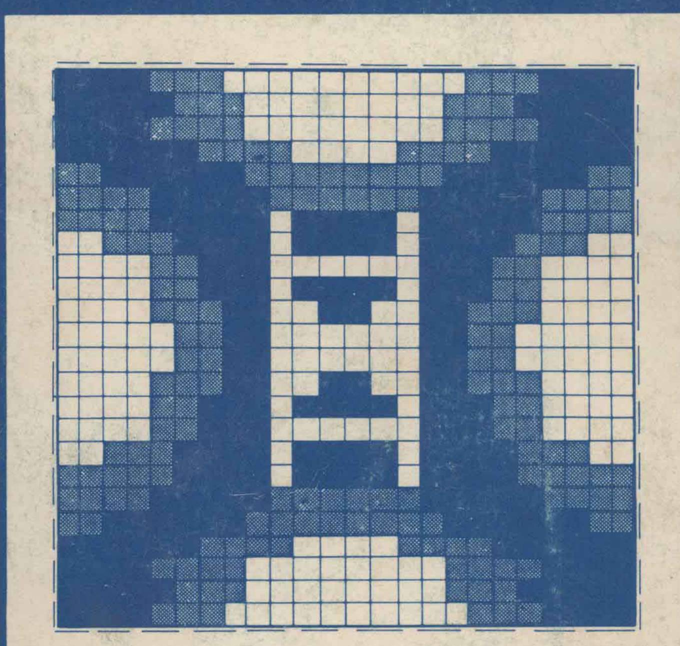
AMD-Vol. 71

Material Nonlinearity in Vibration Problems



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M. SATHYAMOORTHY

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Material Nonlinearity in Vibration Problems

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M. SATHYAMOORTHY
CLARKSON UNIVERSITY

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FOREWORD

Nonlinearities in structural problems can arise in several ways. The material behavior can be nonlinear so that the generalized Hooke's Law is not valid. Alternatively, while the material behavior may be assumed as linear, the strain-displacement relations may become nonlinear due to large deformations. Finite rotations and a number of other factors present in real engineering situations also introduce nonlinearities into structural problems. These problems generally present several difficulties because they are complex and the advantages of uniqueness and superposability of solutions characteristic of problems governed by linear governing equations do not exist in the case of those problems governed by nonlinear equations. As a result, several approximate techniques have so far been used to find acceptable solutions to several nonlinear structural problems. However, several such solutions to nonlinear problems are mainly concerned with geometric-type nonlinearity due to finite displacements or rotations.

A practicing engineer is often called upon to design structures that will be fabricated using new materials or fabrication techniques. These new materials, in many cases, do not exhibit linear material behavior. Recognizing the need within the engineering community for a better understanding of the behavior of structural systems subjected to this type of material nonlinearity, the Shock and Vibration Committee of the Applied Mechanics Division decided in late 1983 to organize a symposium to be held at the 1985 Winter Annual Meeting of the Society. Papers dealing with analytical, numerical as well as experimental approaches were solicited. To sharpen the focus of te symposium, most of the papers were limited to dynamic problems.

This symposium volume, which contains ten papers, deals with a variety od problems in material nonlinearity. They are concerned with bimodular structures behaving dissimilarly in tension and compression, elasto-plastic dynamic behavior of plate and shell structures, response of structures to pulse loading in which plastic deformations in the first phase of response lead to compressive forces in the subsequent motion, material and structural loss factors and the implications of linear and nonlinear material damping, the effects of hysteretic dissipation on the forced vibration resonance motion, probabilistic characterization of a damaged structure, effects of nonlinearity on vibration properties and planar mechanism analysis with material nonlinearity. It is hoped that this publication will provide researchers with a reasonably comprehensive treatment of the progress made in this area as well as generate interest among those researchers who are new to this field.

The editor is grateful to the authors for their contribution and co-operation in preparing this publication, the Chairmen and Vice-Chairmen of the two symposium sessions, the Shock and Vibration Committee and the Applied Mechanics Division for sponsoring this symposium, the Department of Mechanical and Industrial Engineering of Clarkson University for providing facilities to organize the Symposium and to Professor Art Leissa of the Ohio State University for his encouragement and assistance in all phases of the Symposium.

M. Sathyamoorthy
Clarkson University
Potsdam, New York

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SHEAR-BAND INSTABILITY OF NON-ASSOCIATIVE ELASTO-PLASTIC MATERIALS BY STATIONARY WAVE ANALYSIS

J. -G. Beliveau
Department of Civil Engineering
University of Vermont
Burlington, Vermont

H. A. Hassani
Departement de genie civil
Universite de Sherbrooke
Sherbrooke, Quebec, Canada

ABSTRACT

A quasi-steady approach to material instability has recently been proposed for strain rate independent constitutive laws. The matrix equations developed correspond to those obtained by a three-dimensional wave propagation analysis for a stationary wave of an elasto-plastic material. The problem is reformulated to determine at which hardening a physically realistic direction of propagation is obtained to yield a zero eigenvalue of the resultant matrix which incorporates objective stress derivatives. The numerical study confirms previous results for incompressible associative material in plane strain, results for incompressible non-associative behavior also in plane strain, and approximate results for general three-dimensional behavior.

NOMENCLATURE

a_0, a_1, a_2	: coefficients of stability polynomial
[B]	: stability matrix
c	: wave propagation speeds
D	: determinant
E	: Young's modulus of elasticity
$E_{ijk\ell}$	: elasticity tensor
f	: yield function
g	: flow potential for plastic deformation
G	: shear modulus of elasticity
h	: hardening modulus
H	: non-dimensional hardening parameter
I_1	: first invariant of stress tensor
J_2	: second invariant of deviators stress tensor
k	: parameter in yield function
i,j,k,ℓ	: indices
$L_{ijk\ell}$	: tensor relating Cauchy stress rate with Eulerian strain rate
$M_{ijk\ell}$	: tensor incorporating rotation terms
M	: ratio of direction cosines
N	: ratio of direction cosines

n_1, n_2, n_3 : direction cosines of normal to plane wave
s : deviatoric stress tensor
$\dot{t}_{ij}$: intrinsic derivative of nominal stress
t : time
u, v : material displacement and velocity
$\{y\}$: eigenvector

α : parameter in yield function of Drucker-Prager
β : parameter in plastic potential function
γ : scalar multiplier for plastic deformation
ρ : material density
σ : Cauchy stress
$\overset{\nabla}{\sigma}$: Jaumann co-rotational rate of Cauchy stress
$\dot{\sigma}$: material stress rate
ε : strain
μ, μ^* : shear moduli
$d\varepsilon, d\varepsilon^e, d\varepsilon^p$: total, elastic, and plastic strain rate increment
Λ : Lamé constant
λ : eigenvalue
ν : Poisson's coefficient

INTRODUCTION

Conditions for localization of deformation in the form of a shear band have been developed for materials having fairly general constitutive relationships (1,2). Shear band instability criteria for plane strain situations for incompressible materials with an associative flow rate were presented (3) and generalized to non-associative elasto-plasticity of incompressible material (4). There is considerable interest in this phenomenon, particularly with regards to geotechnical materials including soils, where the material is not incompressible, nor does it satisfy the normality rule (5,6,7,8).

Though the analysis presented in these papers is quasi-static, it has long been established that instability considerations for material behavior may be investigated within a dynamic context by considering wave propagation velocities (8,9,10,11). A zero wave velocity corresponds to instability. This is the approach used in this article. Though a general elasto-plastic model is used, strain rate effects are not considered. Only localized instability in the form of a shear band is considered, the basic premise being that, though the state of stress may be determined by particular conditions, such as plane stress, plane strain, uniaxial tension, etc., the material constitutive behavior is characterized in three dimensions, the shear band being a localized phenomenon.

First, we summarize the wave propagation analysis for instability. The two procedures proposed determine critical parameters and the orientation of the normal to the planar wave front. Numerical simulations are compared to previous results limited to incompressible associative material in plane strain (3), incompressible non-associative material also in plane strain (4), and approximate results for general non-associative behavior in three dimensions based on a quasi-static approach (1,2).

INCREMENTAL ELASTO-PLASTICITY

The incremental form of constitutive relations applicable to large strains are

$$\overset{\nabla}{\sigma}_{ij} = L_{ijk\ell}\, d\varepsilon_{k\ell} \tag{1}$$

in which the Eulerian strain rate increment, $d\varepsilon$, is assumed to be a linear sum of elastic, $d\varepsilon^e$, and plastic, $d\varepsilon^p$, components

$$d\varepsilon_{ij} = d\varepsilon^e_{ij} + d\varepsilon^p_{ij} \tag{2}$$

$d\varepsilon$ is defined as

$$d\varepsilon_{ij} = \frac{1}{2} (\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) \tag{3}$$

for v the material velocity. The tensor L corresponds to the Drucker-Prager model in plastic loading (12)

$$L_{ijk\ell} = E_{ijk\ell} - \frac{E_{ijtu} \frac{\partial g}{\partial \sigma_{tu}} \frac{\partial f}{\partial \sigma_{rs}} E_{rsk\ell}}{h + \frac{\partial f}{\partial \sigma_{mn}} E_{mnpq} \frac{\partial g}{\partial \sigma_{pq}}} \tag{4}$$

f, g, and h are the yield function, plastic potential and hardening, respectively

$$f = \sqrt{J_2} + \alpha\, I_1 - k = 0 \tag{5}$$

$$g = \sqrt{J_2} + \beta\, I_1 - k \tag{6}$$

$$h = -\left(\frac{\partial f}{\partial \varepsilon^p_{ij}} + \frac{\partial f}{\partial k} \frac{\partial k}{\partial \varepsilon^p_{ij}} \right) \frac{\partial g}{\partial \sigma_{ij}} \tag{7}$$

$$d\varepsilon^p_{ij} = \gamma \frac{\partial g}{\partial \sigma_{ij}} \tag{8}$$

in which γ is a constant, $\beta = \alpha$ implies associative behavior and

$$J_2 = \frac{1}{2} s_{ij}\, s_{ij} \tag{9}$$

$$I_1 = \sigma_{kk} \tag{10}$$

$$s_{ij} = \sigma_{ij} - \frac{I_1}{3} \delta_{ij} \tag{11}$$

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{s_{ij}}{\sqrt{J_2}} + \alpha\, \delta_{ij} \tag{12}$$

$$\frac{\partial g}{\partial \sigma_{ij}} = \frac{s_{ij}}{\sqrt{J_2}} + \beta\, \delta_{ij} \tag{13}$$

δ_{ij} is the Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \tag{14}$$

WAVE PROPAGATION

The incremental small displacements on large deformations satisfy the equation (10)

$$\frac{\partial \dot{t}_{ij}}{\partial x_j} = \rho \frac{\partial^2 v_i}{\partial t^2} \tag{15}$$

in which ρ is the density and $\dot{t}$ is the intrinsic derivative of t, the nominal stress tensor. For a uniform stress field, it can be shown that the intrinsic derivative of t is related to the material derive of Cauchy stress, $\dot{\sigma}$, (13)

$$\frac{\partial \dot{t}_{ij}}{\partial x_j} = \frac{\partial \dot{\sigma}_{ij}}{\partial x_j} \tag{16}$$

the latter being related to the Jaumann co-rotational rate of stress (1,2)

$$\overset{\nabla}{\sigma}_{ij} = \dot{\sigma}_{ij} + \sigma_{ik}\Omega_{kj} - \Omega_{ik}\sigma_{kj} \tag{17}$$

in which the rotation tensor is given by

$$\Omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \tag{18}$$

The material velocity, v, of a plane wave propagating along a direction given by direction cosines n_1, n_2 and n_3 is assumed to be a function of the independent parameter, ψ

$$\psi = x_1 n_1 + x_2 n_2 + x_3 n_3 - c\ t \tag{19}$$

in which c is the propagation velocity. Introduction of Eqns.(16-19) and (1-14) into Eqn.(15) yields the following eigenvalue problem

$$\left[[B] - \lambda\ [I]\right] \{y\} = \{0\} \tag{20}$$

in which

$$\lambda_i = \rho\ c_i^2 \qquad i = 1,2,3 \tag{21}$$

$$B_{i\ell} = M_{ijk\ell}\ n_j n_k \tag{22}$$

$$M_{ijk\ell} = L_{ijk\ell} + \frac{1}{2}\left(\delta_{j\ell}\sigma_{ik} + \delta_{i\ell}\sigma_{jk} - \delta_{jk}\sigma_{i\ell} - \delta_{ik}\sigma_{j\ell}\right) \tag{23}$$

and

$$y_i = \frac{\partial^2 v_i}{\partial \psi^2} \tag{24}$$

The three eigenvalues depend on the material parameters and the direction of propagation. In the case of a linear elastic material satisfying Hooke's law,

$$E_{ijk\ell} = \Lambda\ \delta_{ij}\delta_{k\ell} + G(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) \tag{25}$$

The three velocities are given by

$$c_1 = \sqrt{\frac{\Lambda + 2G}{\rho}} \tag{26}$$

$$c_2 = c_3 = \sqrt{G/\rho} \tag{27}$$

Λ and G are the Lamé constants related to Young's modulus of elasticity, E , and Poisson's coefficient, ν.

$$G = \frac{E/2}{1 + \nu} \tag{28}$$

$$\Lambda = \frac{E\ \nu}{(1+\nu)(1-2\nu)} \tag{29}$$

For waves propagating in the plane of a plate, satisfying the plane stress assumptions, and for longitudinal waves in a bar, the wave velocities obtained by imposing appropriate boundary conditions are (14)

$$c_4 = \sqrt{\frac{4G\ (\Lambda + G)}{(\Lambda + 2G)\ \rho}} \tag{30}$$

$$c_5 = \sqrt{E/\rho} \tag{31}$$

c_4 is for waves in the plate and c_5 is for longitudinal waves in the bar.

MATERIAL INSTABILITY

For a stationary discontinuity, the determinant of [B] , Eqn.(20), is zero (10), yielding the same result as obtained from quasi-static considerations (1,2). Thus, given a particular direction, instability is defined when at least one of the eigenvalues of [B] is zero. Alternatively, setting the determinant equal to zero yields a polynomial of order six in n_1, n_2 and n_3. The material is locally unstable when one of the solutions to the polynomial yields real values for n_1, n_2, n_3, i.e., loss of ellipticity in a quasi-static approach to localization (15).

For a given loading and stress level and the material constants, G, ν, ρ, α, β and k, the determinant is a function of the three direction cosines and the hardening modulus, h.

Due to the following equality for the direction cosines,

$$n_1^2 + n_2^2 + n_3^2 = 1 \quad (32)$$

the determinant may be considered to be a function of three parameters

$$D(H, N, M) = |B| = 0 \quad (33)$$

in which

$$N = n_1/n_2 \quad (34)$$

$$M = n_3/n_2 \quad (35)$$

$$H = h/G \quad (36)$$

The optimum hardening for instability is then obtained by satisfying Eqn.(33) and (13)

$$\frac{\partial D}{\partial N} = 0 \quad (37)$$

$$\frac{\partial D}{\partial M} = 0 \quad (38)$$

Eqns.(33),(37) and (38) are then three nonlinear equations for the three unknowns N, M, and H.

A second method for determining instability is possible in instances for which M is set equal to zero. In certain applications, there are physical reasons to choose the direction of the perpendicular to the shear band as being in a particular plane, $n_3 = 0$. Eqn.(33) then yields a sixth order polynomial in N

$$N^6 + a_5N^5 + a_4N^4 + a_3N^3 + a_2N^2 + a_1N^1 + a_0 = 0 \quad (39)$$

Instability corresponds to a real root to this polynomial.

NUMERICAL RESULTS

The shear band geometry for $n_3 = 0$ appears in Fig. 1, in which the direction cosines of the unit normal to the shear band are shown. For uniaxial loading in tension of an incompressible associative material in plane strain, real and imaginary portions of the solution to Eqn.(39) are shown in Fig. 2, corresponding to the six roots which, when the material is stable, occur in complex conjugate pairs.

Two of the three eigenvalues of the matrix [B], Eqn.(20), evaluated at the angle of instability corresponding to the real part of N at the critical hardening, here N = 1, are shown in Fig. 3. The third eigenvalue is not shown, due to the incompressibility assumption. The lowest eigenvalue becomes zero at a critical hardening of zero. Finally, stability boundaries (H is hyperbolic or unstable) are shown in Fig. 4, along with parameters determined by a two-dimensional plane strain criterion in which μ and μ^* are defined (3)

$$\overset{\nabla}{\sigma}_{11} - \overset{\nabla}{\sigma}_{22} = 2\mu^* (d\varepsilon_{11} - d\varepsilon_{22}) \quad (40)$$

$$\overset{\nabla}{\sigma}_{12} = 2\mu \; d\varepsilon_{12} \quad (41)$$

as elements of the elasto-plastic tensor. For large hardening modulus, the material is stable and it becomes unstable at h = 0.

The stability boundary for incompressible materials in uniaxial loading ($\sigma_2 = 0$), having a non-associative ($\alpha \neq \beta = 0$) flow rule, are determined by the equation (4)

$$\frac{\mu^*}{\mu} < \frac{1}{2}\left(1 + \frac{\delta\sigma_1}{2\mu}\right) - \sqrt{(1 - \delta^2)\left(1 - \frac{\sigma_1^2}{4\mu^2}\right)} \quad (42)$$

in which the parameter δ is evaluated at the critical hardening ratio, H

$$\delta = \pm \frac{3\,\alpha}{1 + H} \tag{43}$$

The plus and minus sign apply for tension and compression, respectively. For δ equal to zero, Eqn.(42) yields the associative stability boundary shown in Fig.4. Again, for decreasing values of hardening, the material goes from the stable domain to the unstable domain with the transition occuring at the critical hardening parameter, as shown in Fig. 5.

As a third example, an approximate criterion for the critical hardening parameter, H, based on assuming $\overset{\nabla}{\sigma} \approx \dot{\sigma}$, for general three-dimensional behavior of non-associative material, is given by the following relation (1,2)

$$H = \frac{1+\nu}{1-\nu}\,(\beta-\alpha)^2 + \frac{1+\nu}{2}\,(2\,P_2 + \alpha - \beta)^2 \tag{44}$$

in which P_2 is the intermediate principal value of the matrix where elements are given in Eqn.(13).

Results based on Eqn.(39) and on solving Eqns.(33),(37) and (38), for uniaxial tensile loading, confirm this value of critical hardening for instability in both the plane stress and plane strain situations.

The critical hardening ratios H are shown in Fig. 6 for a dilatant material (β=.1, ν=.3) as a function of α. The corresponding direction of the unit normal to the shear band are shown in Fig. 7. Finally, for the case α=.2, the lowest eigenvalue of [B] is shown in Fig. 8, as a function of the hardening ratio for both the plane stress and plane strain situations, in which N was set equal to the corresponding value at critical hardening. The zero eigenvalue occurs at the critical hardening ratio in both instances.

CONCLUSION

Local material instability may be investigated within a wave propagation framework in three dimensions. The instability criterion obtained by considering stationary waves is identical to that obtained from quasi-static considerations. The numerical investigation reported in this article confirms previous stability results of incompressible material behavior in plane strain, both for associative and non-associative elasto-plastic formulations and an approximate formula for general three-dimensional material behavior with arbitrary loading.

ACKNOWLEDGEMENTS

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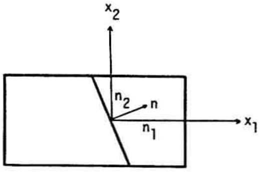

Fig. 1: Shear band geometry in a plane ($n_3 = 0$)

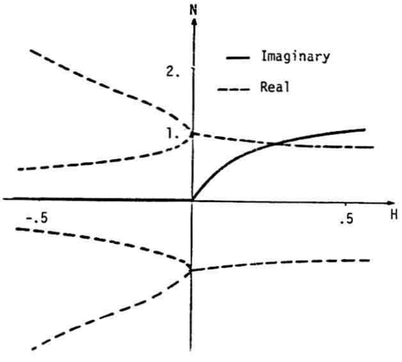

Fig. 2: Real and imaginary portion of N for plane strain behavior of associative incompressible elastoplastic material under uniaxial loading ($\nu = .5$, $\alpha = \beta = 0$)

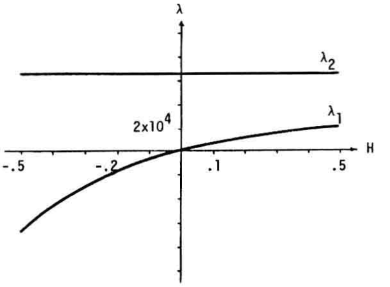

Fig. 3: Eigenvalues for problem in Fig. 2 with N = 1

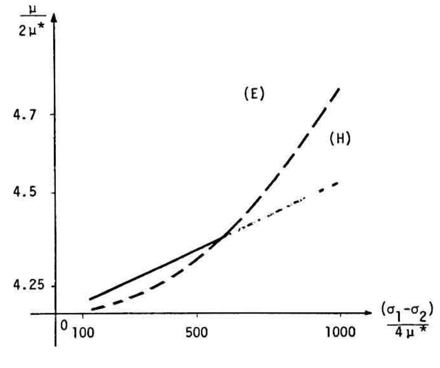

Fig. 4: Stability boundary --- and results —— for problem in Fig. 2

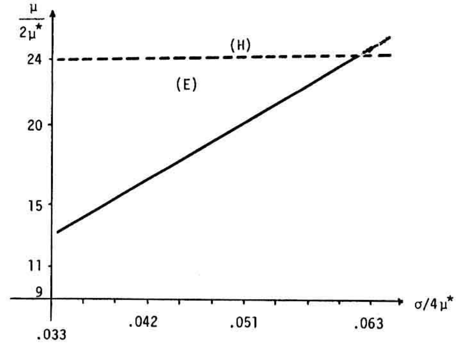

Fig. 5: Stability boundary at critical hardening ratio and results for non-associative incompressible material under uniaxial loading (ν=.5, β=0, α=.1)

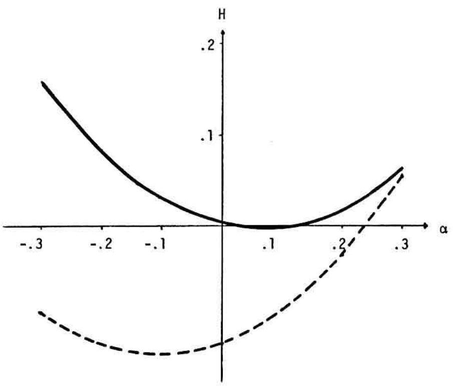

Fig. 6: Critical hardening ratio in uniaxial tension (ν=.3, β=.1)
—— plane strain --- plane stress

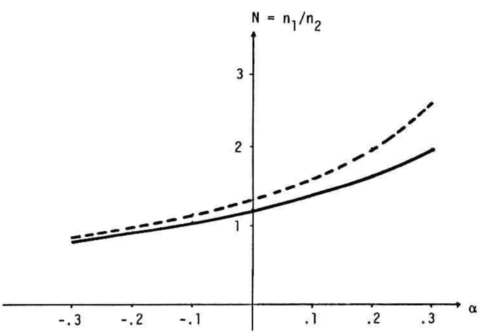

Fig. 7: Critical (N) in uniaxial tension (ν=.3, β=.1)
—— plane strain --- plane stress

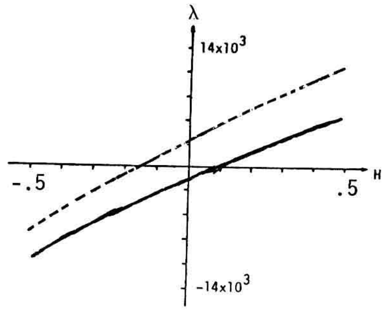

Fig. 8: Lowest eigenvalue of [B] at critical N for uniaxial loading in plane strain —— and plane stress --- (ν=.3, α=.2, β=.1)

N plane strain = 1.63 N plane stress = 1.95

DAMAGE DIAGNOSIS FOR MDF SYSTEM WITH MATERIAL NONLINEARITY

C. F. Chang, Research Associate and **F. D. Ju**, Professor
Department of Mechanical Engineering
University of New Mexico
Albuquerque, New Mexico

T. L. Paez
Sandia National Laboratories
Albuquerque, New Mexico

ABSTRACT - The present paper established a procedure for the probabilistic characterization of a damaged structure, which is modeled as a nonlinear MDF system. The random excitation may be either stationary or nonstationary. The stiffness matrix is nonlinear to simulate the elastoplastic behavior of a damaged structure. The stiffness matrix is also random to characterize the material and environmental variations. The governing stochastic differential equation is resolved into one for the mean response and another for its random component. Responses, their statistical moments and cross-moments are solved with discrete-time recurrence formulations. The probability of structural damage or the structural reliability, is than estimated by the upper bound of the cumulative energy dissipation. The formalism of approach in formulating the solutions for a generic class of MDF nonlinear problem with Prandt-Reuss material permits ready adaptation to FEM analysis.

NOMENCLATURE

$\mathbf{A}_1$, $\mathbf{A}_{2j}$, $\mathbf{A}_3$	Constant square matrices
$\mathbf{b}_i$	Modal shape vector
$E[\cdot]$	Mean value of $[\cdot]$
E_s, E_v,	Energy dissipation in element and beam
$\mathbf{f}(t)$, $\boldsymbol{\varphi}(t)$, $\mathbf{F}(t)$	Excitation, its mean and random component
$\mathbf{k}(z)$, $\boldsymbol{\lambda}(\mu)$, $\mathbf{K}$	Stiffness matrix, its mean and random component
$\mathbf{m}$, $\mathbf{c}$	Constant mass and viscous matrices
$\mathbf{R}(\mu)$	Nonlinear restoring force vector
$\mathbf{z}(t)$, $\boldsymbol{\mu}(t)$, $\mathbf{Z}(t)$	Response, its mean and random component
$\dot{\mathbf{z}}(t)$, $\dot{\boldsymbol{\mu}}(t)$, $\mathbf{V}(t)$	Velocity, its mean and random component
$\ddot{\mathbf{z}}(t)$, $\ddot{\boldsymbol{\mu}}(t)$, $\mathbf{A}(t)$	Acceleration, its mean and random component
α, γ	Coefficients of Rayleigh damping
β	Coefficient of variation of Young's Modulus
ε_t	Total strain
ε_o	Permanent set
σ_t	Total stress
σ_y	Yield stress
ζ_i, $\mu_{\zeta i}$	Random parameter and its mean

1.0 INTRODUCTION

In structural damage diagnosis, the portion of structure where the material is at the damage state will be inelastic, resulting in nonlinear stiffness, especially when the structure is subjected to some extreme excittation. For intensive shock loading the excitation is essentially random. Furthermore, the stiffness will display certain random characteristics, which arise principally from structural assembly defects, either in fabrication or in aging, and from the material properties especially in the nonlinear range. The statistical properties of the response, therefore, must be taken into consideration. In the present paper, a generic structure is to be modeled by a MDF (Multi-Degree of Freedom) system of which the stiffness matrix is nonlinear and random. The mass and damping matrices are for most practical cases constant and deterministic. The objective of the present paper is to establish a procedure for the probabilistic characterization of the inelastic MDF system and its response due to a random excitation. Based on the results, the damage and the reliability of the structure can be assessed.

2.0 FORMULATION

The governing differential equation of motion for a nonlinear structural framework, modeled as a discrete MDF system, can be written as

$$\mathbf{m}\,\ddot{\mathbf{z}} + \mathbf{c}\,\dot{\mathbf{z}} + \mathbf{k}(\mathbf{z})\,\mathbf{z} = \mathbf{f} \tag{1}$$

where $\mathbf{m}$, $\mathbf{c}$, $\mathbf{k}(\mathbf{z})$ are the NxN mass, damping, and stiffness matrices, $\mathbf{f}$ is the external load vector; $\mathbf{z}$, $\mathbf{z}$, $\mathbf{z}$, are the displacement, velocity, and acceleration vectors of the system. In this investigation, $\mathbf{m},\mathbf{c}$ are assumed to be deterministic and constant. The stiffness is represented by a matrix of random variable which may correlate to the response $\mathbf{z}(t)$. The quantities $\mathbf{f}$, $\mathbf{k}(\mathbf{z})$, and $\mathbf{z}$ can be resolved as follows

$$\mathbf{f} = \boldsymbol{\varphi} + \mathbf{F}, \quad \mathbf{k}(\mathbf{z}) = \boldsymbol{\lambda}(\boldsymbol{\mu}) + \mathbf{K}, \quad \mathbf{z} = \boldsymbol{\mu} + \mathbf{Z} \tag{2}$$

where $\boldsymbol{\varphi}$, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are mean values that

$$E[\mathbf{f}] = \boldsymbol{\varphi}, \qquad E[\mathbf{k}(\mathbf{z})] = \boldsymbol{\lambda}(\boldsymbol{\mu}), \qquad E[\mathbf{z}] = \boldsymbol{\mu} \tag{3}$$

It is noted that, from (2), the nonlinear properties of stiffness are a assumed to be reflected by its mean component. The randomness of stiffness is represented by $\mathbf{K}$ which is a matrix of random variables. $\mathbf{F}$ is a non-stationary band-limited white noise. Also it is noted that the random quantities, $\mathbf{F}$, $\mathbf{K}$ and $\mathbf{Z}$ introduced in (2) are all zero mean. Substitution of (2) into (1) yields

$$\mathbf{m}(\ddot{\boldsymbol{\mu}}+\ddot{\mathbf{Z}}) + \mathbf{c}(\dot{\boldsymbol{\mu}}+\dot{\mathbf{Z}}) + \left(\boldsymbol{\lambda}(\boldsymbol{\mu})+\mathbf{k}\right)(\boldsymbol{\mu}+\mathbf{Z}) = \boldsymbol{\varphi} + \mathbf{F} \tag{4}$$

Taking expectation on both sides of the above equation results in the mean value equation,

$$\mathbf{m}\ddot{\boldsymbol{\mu}} + \mathbf{c}\dot{\boldsymbol{\mu}} + \boldsymbol{\lambda}(\boldsymbol{\mu})\boldsymbol{\mu} = \boldsymbol{\varphi} - E[\mathbf{k}\mathbf{z}] \tag{5}$$

The difference of (5) and (4) yields the equation for the random component.

$$\mathbf{m}\ddot{\mathbf{Z}} + \mathbf{c}\dot{\mathbf{Z}} + \boldsymbol{\lambda}(\boldsymbol{\mu})\mathbf{Z} = \mathbf{F} + E[\mathbf{K}\mathbf{Z}] - \mathbf{K}\mathbf{Z} - \mathbf{K}\boldsymbol{\mu} \tag{6}$$

In (5) and (6), $\mathbf{KZ}$ and $E[\mathbf{KZ}]$ being the product of two random quantities, are higher order terms. By neglecting these terms, (5) and (6) are reduced to the following expressions

$$\mathbf{m}\ddot{\boldsymbol{\mu}} + \mathbf{c}\dot{\boldsymbol{\mu}} + \mathbf{R}(\boldsymbol{\mu}) = \boldsymbol{\varphi} \tag{7}$$

$$\mathbf{m}\ddot{\mathbf{Z}} + \mathbf{c}\dot{\mathbf{Z}} + \boldsymbol{\lambda}(\boldsymbol{\mu})\mathbf{Z} = \mathbf{F} - \mathbf{K}\boldsymbol{\mu} \tag{8}$$

The omission of the higher order terms are postulated for the MDF systems. For SDF systems, the errors in response and its statistics resulting from omission of the higher order terms will be discussed in Sec. 5.0.

3.0 NONLINEAR MODEL

The present paper will treat a generic class of elasto-plastic material that satisfies the Prandtl-Reuss relationship, Fig. 1. For a beam with symmetric cross section, the moment for any section at location x, measured longitudinally along the beam, can be evaluated by

$$M(x) = \int_A \sigma_t \, w(h)h \, dh \tag{9}$$

where w(h) is the width of the beam at the distance h measured from neutral axis and σ_t is the stress at the same location. From Figure 1, σ_t can be replaced by

$$\sigma_t(x,h) = [\varepsilon_a(x) + \varepsilon_b(x,h) - \varepsilon_o(x,h)]E \tag{10}$$

where $\varepsilon_a(x)$, $\varepsilon_b(x,h)$ are axial and the bending strain, respectively, and $\varepsilon_o(x,h)$ is the permanent set at locations x and h. For pure bending, the deflections in the beam are small, (9) becomes

$$M(x) = EIy'' - E\int_{-c}^{c} \varepsilon_o(x,h) \, hw(h) \, dh \tag{11}$$

where E is the Young's modulus, I is the moment of inertia of the cross section, y" is the second derivative of the deflection with respect to the coordinate x, and c is the half depth of the symmetric cross section. In the present paper, all external load, without loss of generality, are resolved at nodal point. Hence

$$EIy' = E\int_0^x du \int_{-c}^{c} \varepsilon_o(u,h)h \, w(h)dh + \frac{1}{2} C_1x^2 + C_2x + C_3 \tag{12}$$

$$EIy = E\int_0^x dv \int_0^v du \int_{-c}^{c} \varepsilon_o(u,h)h \, w(h)dh + \frac{1}{6} C_1x^3 + \frac{1}{2} C_2x^2 + C_3x + C_4 \tag{13}$$

The constant C_i, i = 1,..,4 are prescribed from boundary conditions. The axial strain can be evaluated with analogous approach.

$$\varepsilon_a(x) = \frac{1}{EA}\left(P + E\int_{-c}^{c} \varepsilon_o(w,h) \, w(h) \, dh\right) \tag{14}$$

where P is the axial forces, L is the length of the beam, A is the cross-sectional area. The permanent sets are yet to be determined. For that purpose an iteration scheme is developed using finite difference method described as follows in five steps:

1. The displacement at time t_{j+1} can be evaluated from (7) by using central difference assumption;

$$\mu_{j+1} = A_1^{-1}[2m\mu_j + A_3\mu_{j-1} + \Delta t^2(\varphi_j - R(\mu_j))] \tag{15}$$

where

$$A_1 = \frac{1}{2} c \, \Delta t + m \qquad A_3 = \frac{1}{2} c \, \Delta t - m \tag{16}$$

It is assumed that the system starts at rest.

2. For each beam member, the nodal displacements can be obtained from μ_{j+1} which are in global coordinates. Hence, the coordinate transformation is required in order to obtain the deformations of each beam member in local coordinates.

3. At the beginning of time t_{j+1}, it is designated that the permanent sets for this time step are equal to those permanent sets at time t_j. Terms such as y" and $\varepsilon_o(x)$ can subsequently be computed. The total strain $\varepsilon_t(x)$ is then computed.

4. Based on $\varepsilon_t(x)$ new permanent sets are evaluated, with which new estimates of y, $\varepsilon_o(x)$ and ε_t are evaluated. Steps 3 and 4 are iterated until convergence of the values of permanent sets.

5. From the final permanent sets, the restoring forces $\mathbf{R}(\mu_{j+1})$ for each member can be evaluated. Accordingly, the global restoring forces $\mathbf{R}(\mu_{j+1})$ are assembled with the restoring forces of each beam member.

In the above iteration scheme, the neutral axis does not change throughout the computation. Because of this assumption, the permanent set will first converge, then alternate between two values. In such cases, an approximation for the permanent sets can be established by averaging the two values.

4.0 PROBABILISTIC SOLUTION OF MDF, NONLINEAR SYSTEM

Since (8) characterizes the random component of the structural response, the displacement response at time t_{j+1} can be solved by using central difference approximation. Namely,

$$Z_{j+1} = A_1^{-1}(A_{2j} \, Z_j + A_3 Z_{j-1} + \Delta t^2 F_j - \Delta t^2 K \mu_j) \tag{17}$$

where A_1 and A_3 are given by (16) and

$$A_{2j} = 2m - \Delta t^2 \lambda(\mu_j) \tag{18}$$

in which $\lambda(\mu_j)$ is the equivalent stiffness matrix at time t_j; the sth column in $\lambda(\mu_j)$ is by definition

$$\lambda_s(\mu_j) = \frac{\partial R(\mu_j)}{\partial \mu_s} \tag{19}$$

The response covariance matrix [1] $E[Z_{j+1} Z_{j+1}^T]$, can then be established.

It is noted that F_j is independent of Z_{j-1}^T if the excitation is a sequence of independent and independently arriving random impulses. It is reasonable to assume that F_j is independent of K. Also, F_j is independent of Z_j^T if the input is white noise type excitation. Accordingly, it is possible to show that