Dimensional analysis for engineers

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Preface

So MUCH has been written on the subject of dimensional analysis that one wonders why anyone considers it desirable to add to the literature on the subject. The following quotation is from the preface to the revised edition of the classic work of Bridgman (1931):

> Since the first printing of the book, I have observed to my great surprise that in spite of what seemed to me a lucid and convincing exposition there are still differences in fundamental points of view, so that the subject cannot yet be regarded as entirely removed from the realm of controversy.

Many subsequent books have been written on this subject, but the quotation appears as true today as it was over forty years ago, and the present author has been unable to suppress a desire to try his hand at the problem of reducing the controversial element. This desire has been strengthened by long, direct experience in the use of dimensional analysis as a powerful aid in making the rapid technical decisions which are required of an engineer. The author hopes that his somewhat different approach may clarify the applications and limitations of dimensional analysis.

For the amount of time and effort required to understand it and to use it, dimensional analysis offers unusually great rewards and it therefore should become a part of the tool kit of every engineer—often the first tool to be applied to a new problem.

At present, dimensional analysis is not commonly used in new situations, even by advanced engineering students. In fact there is an increasing number of graduates escaping from technical schools having virtually no facility with this tool. This situation is due, at least in part, to a general failure of technical schools to introduce the student to dimensional analysis early enough. While this book is primarily addressed to teachers, advanced students, and practising engineers, students with an introductory knowledge of physics should find much of it comprehensible and useful. It is hoped that teachers of elementary subjects will be inspired to introduce their students to the simpler ideas of dimensional analysis. While the benefits of using dimensional analysis in statics, for example, are not generally as large as those in fluid mechanics, statics offers an early starting point which can be easily understood, and the rewards are well worth the effort. As the student progresses to dynamics, the advantages of the dimensional approach become more obvious and, if these advantages are kept before the student, by the time he arrives at the study of fluid mechanics, heat transfer, and elasticity, where the reward is very high, the dimensional approach should have become almost automatic. For advanced students the author can only hope that they will experience some of the excitement and fascination which he himself has felt in applying dimensional analysis to real problems.

The basis of dimensional analysis is usually considered to be the Pi theorem of Buckingham (1914). An improved proof by Bridgman (1931) will not be repeated here, but a statement of the theorem (essentially that of Bridgman) will be found in the appendix. Application of the theorem leads to the solution of a number of simultaneous linear equations. In order that the theorem shall correctly state the number of resulting independent dimensionless groups, these equations must be independent. Solving these equations and confirming their independence by a matrix method is a somewhat time consuming process. Most engineers find short cuts to determine the dimensionless groups. If the short cut used does not reveal whether the equations were indeed independent, the number of groups sometimes mysteriously turns out to be larger than predicted by the theorem.† A simple and rapid routine is presented which avoids this difficulty.

There are other situations, unconnected with the Pi theorem, which sometimes result in a number of dimensionless groups which, although it may be equal to the number predicted by the Pi theorem, is larger than is necessary to specify the problem completely. A different approach to the subject makes the origin of these difficulties clear and indicates how to avoid them.

Because of difficulties such as these, some engineers prefer postponing the use of dimensional analysis until the governing equations and the boundary conditions have been formulated. If one *can* write the governing equations and the boundary conditions, the application of dimensional analysis may then give a more specific and therefore more useful statement of the form of the solution than if one can only state the pertinent physical laws. Why, then, not wait to apply dimensional analysis until after the governing equations have been formulated? In many problems, writing the governing equations and boundary conditions is not possible even though the pertinent physical laws are well known. In such cases, dimensional analysis and experiment may be the only methods of attack available. Moreover, there is much physical insight to be gained by the application of dimensional analysis early in the process of learning about any particular problem.

The author has concentrated on application of the method *before* the governing equations are available and has tried to chart some of the more dangerous shoals on which he himself has run aground. Any arbitrary rule such as 'use dimensional analysis first' tends to make a game of what perhaps should be strictly business. Dimensional analysis is a tool to be used in conjunction with any other tools in any order appropriate to the solution of any particular problem. However, games are sometimes fun and often instructive. The author has been unable to resist a paragraph or two to suggest a new

† See pp. 79–80, 121–3.

game—the application of dimensional analysis to mathematical formulation in the absence of a specific physical problem. This game appears at the end of Chapter 5.

Many of the examples which will be used are problems in which the governing equations can easily be written and solved. Such examples have purposely been chosen so that a check of the result is readily available. However, the method is clearly most useful in problems which are too complex to allow such analytical solution or of such a general character that their analytical solution would require an overwhelming amount of information.

Further, it is possible to treat problems of a more general nature which cannot be completely analysed because they are insufficiently specified. Many valuable generalizations can result from dimensional analysis of such problems. Some of these generalizations are discussed in Chapter 5.

In working out examples, the author has often found himself diverted by interest in the example and the rather extraordinary amount of generally useful information that can sometimes be obtained from a specific problem. While an effort has been made to resist digression, it is hoped that the reader will excuse the author if at times he seems to be expounding the subjects of elasticity, vibration, heat transfer, etc. rather than strictly tending to the business of dimensional analysis. Naturally, examples have been chosen from the author's experience. The paucity of examples from certain fields reveals the author's inexperience in those areas. It is hoped that the reader will be able to supply examples of his own to make up for the author's deficiencies.

January 1974 E. S. T.

Acknowledgement

THE evolutionary process which led up to this book has been going on for a period of more than thirty years. Countless discussions with students, colleagues, and other friends have helped to educate the author, who is indeed grateful to these people for their patience in putting up with his often rigid attitude. The fact that the author's basic approach has gone through at least two major upheavals is proof that these discussions were not without effect.

Special mention should be made of Professor Howard W. Emmons of Harvard, whose perceptive comments wrought much havoc in early drafts and are responsible for the present approach to the subject.

Professors Robert L. Halfman, Frank A. McClintock, and Stephen H. Crandall of M.I.T. and especially Professor Reinout P. Kroon of University of Pennsylvania also gave the time and thought required to make helpful suggestions. Dr. Charles H. Blake, formerly of M.I.T., contributed to the biological information and guided the author to interesting and pertinent reading.

Professor Mayo D. Hersey, whose own contributions to dimensional analysis are considerable, scrutinized the manuscript and helped to remove errors large and small and to clarify incomprehensible language.

No work on dimensional analysis can be uninfluenced by the work of the late Percy W. Bridgman of Harvard, whose book will always remain a classic.

Mention must also go to the author's wife, who managed to remain quiet as a mouse for what must have seemed interminable periods while the author doodled, chewed pencils, cursed, and produced quantities of waste paper.

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Contents

1. Introduction: definitions: the fundamental process

DIMENSIONAL analysis is used, often unconsciously, by practically every engineer or scientist who deals with physical problems.

In addition to providing a guide to experimental planning and to the correlation of data, it sometimes offers an aid to the solution of physically-based differential equations. To the engineer, perhaps its most important use is as a means of developing the ability to generalize from experience and thus to apply knowledge to a new situation. Although always perilous, generalization is essential to bring an element of order into an otherwise chaotic world.

Dimensional analysis is concerned with the nature of the relationship between the various quantities which enter a physical problem. Before dimensional analysis can be applied it must be known that one and only one relationship exists between a certain number of physical quantities, and that no pertinent quantities have been omitted, and no extraneous quantities included.

Such a relationship can be expressed by the symbolism:

$$\phi(q_1, q_2, q_3, \dots q_n) = 0, \tag{1.1}$$

where the qs are the numerical values of all the quantities which are pertinent to the problem.

The process of dimensional analysis is one of grouping the original quantities into 'dimensionless ratios' Π to form a new relationship

$$\phi(\Pi_1, \Pi_2, \Pi_3, \dots \Pi_m) = 0 \tag{1.2}$$

which contains all of the information pertinent to (1.1).

In general $m < n$ and thus relation (1.2) is more specific than (1.1). Since it is more specific, the form (1.2) contains either more information pertinent to the problem or less extraneous information than the original relation (1.1).

It will be shown that the latter is the case, that is, *dimensional analysis is a process for eliminating extraneous information from a relation between quantities*. Looked upon in this way, many of the difficulties associated with the philosophy and processes of dimensional analysis turn out to be the result of concealed pathways through which extraneous information creeps into a problem. Furthermore, in certain problems it is possible to remove additional extraneous information beyond what can be accomplished by the usual method. As another bonus, this approach illuminates the fallacy of applying dimensional analysis to a problem which is insufficiently understood (for example, one in which even the physical laws involved are unknown). Attempts to apply dimensional analysis in such cases give the subject the reputation of being a black art, which in these circumstances it is indeed. In such cases dimensional analysis approaches the clearly irrational process of trying to remove extraneous information from no information at all.

Dimensional analysis is a step toward the goal of describing a physical entity or phenomenon in terms of relationships between numbers. The ultimate goal can never be reached by dimensional analysis alone, and it is therefore not a substitute for complete analysis or for experiment. It is nevertheless a very useful tool, especially for improving physical insight. It can sometimes give useful results which are virtually impossible to obtain by any other method.

In order to progress toward the goal it is necessary to have rules for relating numbers to the problem at hand. For this purpose some definitions are essential.

Definitions

A *physical quantity* is a concept such as time, mass, temperature, velocity, etc. which can be expressed numerically in terms of one or more standards.

Not all physical concepts can be numerically expressed, consequently not all physical concepts are physical quantities (for example, an electron is certainly a physical concept but not a physical quantity, although the mass of an electron and its electric charge are, of course, physical quantities).

Fourier (1822), who used dimensional analysis long before it was formally recognized, conceived of a physical quantity as having two characteristics: a *numerical measure* and a *concept.* A football field (in the U.S.A.) is 300 ft long. The numerical measure is 300 and the concept is distance or length.

The word *concept* calls up a mental image which depends essentially upon the rules for making a measurement. Thus the concept of length evokes the mental image of standardized rods laid end to end alongside the length to be measured. For various reasons the actual method used may not be the one from which the mental image derives. For example, for measurement of very large or very small distances, the physical addition of rods is inconvenient, sometimes to the point of impossibility. We demand that whatever substitute we use for addition of measuring rods be shown to give the same results over at least some range of distances, and thus we have a basis (logically less than perfect) for calling the quantity found by the substitute method 'length'.

The words *numerical measure* imply a comparison in accordance with the rules of measurement of the quantity in question with some sort of *physical standard.*

Standards have been adopted essentially for purposes of communication. They make it possible for two physical quantities to be compared without the necessity of bringing the corresponding physical entities together. This is done by the introduction of a third entity—the physical standard. Comparison of two physical quantities can be made by comparing each with the physical standard. When communication is direct, we can often get along without physical standards. Instead of saying 'cut it 34 inches long' it is possible to say 'make it fit here'.

By their very nature, physical standards are *arbitrary* and generally unrelated to the problem in hand. Other than convenience in reproduction, it really does not much matter whether the standard of length was derived from measurement of a king's foot, a fraction of the earth's circumference or a wavelength of light from a certain line in the spectrum of Krypton 84. However, it is important that the *concept* of length as determined from the rules for its measurement remain the same for any of these standards.

Units are derived from standards. The fact that the standard of length is now the wavelength of a certain spectral line does not, of course, mean that this wavelength must be used as a unit. We may still use metres, feet, or whatever we choose. A unit is, thus, some arbitrary fraction or multiple of a standard. Standards are changed as time goes on and technology becomes more sophisticated. Originally a fraction of the earth's circumference, the metre is now officially defined as a multiple of a wavelength of light.

The magnitude of the numerical measure depends on the unit or units of measurement. We may change the measuring unit from feet to yards, thereby changing the numerical measure of the length of a football field from 300 to 100 but leaving the concept of distance, and of course the actual physical situation, unaltered. Note that when the size of the unit of length is changed by a factor (in this case three), every number representing a numerical measure of length will be changed by the reciprocal of the same factor. This relationship is true only of a certain class of physical quantities which will be called *linear*. Most common physical quantities are linear.

Linear physical quantities

Of all physical quantities, only those which can be said to have a 'linear scale' can be used in dimensional analysis. Bridgman (1931, 1945) states this restriction by limiting the application of dimensional analysis to those quantities which have the property of 'absolute significance of relative magnitude'. The phrases 'twice as fast' and 'twice as far' express relative magnitudes and have clear quantitative meanings. An interval of one unit anywhere along the scale of such quantities is, in some sense, equivalent to one anywhere else in the scale.

Physical quantities of another class have arbitrary scales: for example, Moh's scale of hardness (1 talc; 2 gypsum; 3 calcite; 4 fluorite; 5 apatite; 6 feldspar; 7 quartz; 8 topaz; 9 sapphire; 10 diamond). Similar examples are the Beaufort scale of wind velocity and the International scale of roughness of the sea. Such quantities can be used to express equalities and inequalities (diamond is harder than sapphire: $10 > 9$) but are otherwise unsuited either to numerical computation or to dimensional analysis. Certainly there is no meaning to the statement that 'gypsum is twice as hard as talc' ($2 = 2 \times 1$).

Special care is necessary to ensure the validity of dimensional analysis when dealing with temperature. Twice as hot, if it means more than 'much hotter' does so only in special circumstances. The peculiarity of the temperature scale seems to be due to the fact that addition of the temperature of body a to that of body b has no physical meaning; thus the usual method of constructing a scale by adding units (as in length or mass) is inappropriate.

In order to determine if a physical quantity is linear it is sufficient to ask if the arithmetical operation of addition has a meaningful physical counterpart. By the very nature of the measurement of length, for example, it is clear that addition has such a meaning and that length is indeed a linear physical quantity. Similarly it is clear that force, mass, and time are linear quantities in the realm of ordinary engineering problems. If we were to consider problems involving velocities of the order of the velocity of light, a closer examination of the linearity of these quantities would be in order (is there any meaning to the phrase 'twice as fast as the velocity of light'?).

Primary and derived quantities

It is conventional to look upon certain quantities, for example, length and time, as 'primary' or 'fundamental' and other quantities, such as velocity, acceleration, area, and volume, as 'derived'. As will appear later, the choice of primary quantities can be made arbitrarily. The fact that a physical quantity is designated as a primary quantity merely means that a unit of measurement *can* be assigned to it, *independent* of the units of measurement chosen for the other primary quantities involved in the problem. For example, if length and time are chosen as primary quantities, velocity can be expressed as length per unit time, acceleration as length per unit (time $\times$ time), area as $(\text{length})^2$, etc. On the other hand, it would be equally logical to consider velocity V and time T as primary quantities and length, area, volume, and acceleration as derived units equal to VT, $(VT)^2$, $(VT)^3$, (VT^{-1}), respectively. Perhaps the reason we usually look upon velocity as a derived rather than as a primary quantity is that velocity has usually been determined by measuring a time and a length and performing a calculation. However, the Doppler-shift radar technique used by the police to measure the speed of motor cars is a direct method of measuring velocity without any distance measurement whatever.† If we did most of our measuring this way it would be more natural to consider time and velocity as primary quantities and length as a derived quantity.

In a given problem it is necessary that there be a sufficient number of

† Presumably, a distance must have been measured in establishing the velocity of light, but this is necessary only to reduce the result to familiar units. If we were content to express speed as a fraction of the velocity of light, no such calibration would be necessary. (You are arrested for driving at one ten millionth the velocity of light—67 m.p.h.!)

primary quantities so that each of the derived quantities can be expressed in terms of products of these primary quantities.

For purposes of dimensional analysis it is most inconvenient to regard primary quantities as rigidly imposed. It will appear that each problem, or at least each class of problems, is best handled by a specially selected set of primary quantities. The word 'fundamental', often applied to primary quantities, implies an undesirable rigidity and has been avoided for this reason.

Dimensions

The word *dimension* refers to the relationship of a derived quantity to whatever primary quantities have been selected. We say that the dimensions of area are (length)^2 or the dimensions of velocity are length/time. These statements are written:

$$[A] = \mathrm{L}^2$$
$$[V] = \mathrm{LT}^{-1}.$$

For [ ] read 'the dimension(s) of'. For = read 'is (are)'. Primary quantities are conventionally placed on the right-hand side of these 'equations'. Thus the above symbolic shorthand implies that length and time have been selected as primary quantities while area and velocity have been considered as derived quantities.

The manner in which the numerical measure of a quantity changes with changes in the size of measuring units can be used to determine the *dimensions* of the quantity. The rule for testing the dimensions of any physical quantity is as follows. Change the size of measuring unit of one of the primary quantities by a factor k. The numerical measure of the physical quantity will change by a factor k^{-n}. The primary quantity then appears in the physical quantity to the nth power. Repeating this process for each primary quantity in turn will determine the dimensions of the physical quantity. In the example of the football field a change in the length unit from feet to yards ($k = 3$) resulted in a change in the numerical measure of the length of the field by a factor of $\frac{1}{3}$—from 300 to 100 or k^{-1}. Therefore, in this case $n = 1$. Changes in the size of the units of mass, force, time, etc. cause no change in the numerical measure, and we may conclude that the dimensions of the quantity we have measured are L^{+1}.

If we agree to derive the unit of area from the unit of length (square feet, square yards) then the numerical measure of the area of the field would change by a factor of 9 or k^{-2}, and we would conclude that area has the dimensions of (length $\times$ length) or

$$[A] = \mathrm{L}^2.$$

While the procedure in these particular cases is trivial, it provides an essential test for settling disputes as to the dimensions of any quantity in more complex situations.

We see that the statement

$$[A] = L^2$$

implies that the selection of a unit of length is necessary and sufficient to determine the numerical measure of an area, while the statement

$$[V] = LT^{-1}$$

implies that selection of a unit of length and one of time are necessary and sufficient to determine the numerical measure of a velocity.

The dimensions of any quantity depend upon the choice of primary quantities. In fact, it will subsequently become clear that it is possible for a given quantity to appear as dimensionless with one choice of primary quantities and dimensional with another choice.

Dimensionless quantities

It follows that the test of a dimensionless quantity is that its numerical measure is unchanged by *any* change in the size of the *extraneous* measuring units.

It is sometimes argued that the length of an object measured in metres is dimensionless, since that number represents the ratio of two lengths—that of the object to that of the standard metre bar. The crucial point is whether the size of the measuring unit is derived from a standard which is *extraneous and independent of the particular physical situation* or to some standard *derivable from the problem*. Pressure cannot be made a dimensionless quantity by expressing it in atmospheres unless the standard atmospheric pressure is physically pertinent to the problem under investigation. As an example, the instantaneous barometric pressure at sea-level in the vicinity of Greenwich Observatory is continuously recorded. If the instantaneous values were to be divided by the mean value, one would certainly consider the result to be dimensionless, although the result of such division would merely be to express pressure in atmospheres instead of in some other unit. However, in this case, the standard is indeed derivable from the problem and is *not*, therefore, an *extraneous* unit. Furthermore, lengths expressed in 'feet' might be dimensionless if the investigation concerned the physical proportions of King Henry VIII, whose foot was used as a standard! Inescapably, the question as to whether a quantity is dimensionless or not depends upon the particular problem under discussion.

An example of a quantity which is almost universally considered dimensionless (mass taken as a primary quantity) is atomic weight.† The atomic weight of hydrogen is 1·008 whether mass is measured in slugs, kilograms, or stone. By definition, atomic weight is the mass of an atom divided by one-twelfth of the mass of an atom of carbon 12.

† The word 'weight' is an unfortunate precedent. Clearly it should be atomic *mass*.