D. J. H. GARLING

A COURSE IN Mathematical Analysis

VOLUME I

Foundations and Elementary Real Analysis

A COURSE IN MATHEMATICAL ANALYSIS

Volume I
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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town,
Singapore, São Paulo, Delhi, Mexico City

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9781107032026

First published 2013

Printed and bound by CPI Group (UK) Ltd, Croydon CR0 4YY

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data
Garling, D. J. H.
Foundations and elementary real analysis / D. J. H. Garling.
pages cm. – (A course in mathematical analysis; volume 1)
Includes bibliographical references and index.
ISBN 978-1-107-03202-6 (hardback) – ISBN 978-1-107-61418-5 (paperback)
1. Mathematical analysis. I. Title.
QA300.G276 2013
515–dc23 2012044420

ISBN 978-1-107-03202-6 Hardback
ISBN 978-1-107-61418-5 Paperback

A COURSE IN MATHEMATICAL ANALYSIS

Volume I: Foundations and Elementary Real Analysis

The three volumes of *A Course in Mathematical Analysis* provide a full and detailed account of all those elements of real and complex analysis that an undergraduate mathematics student can expect to encounter in the first two or three years of study. Containing hundreds of exercises, examples and applications, these books will become an invaluable resource for both students and instructors.

Volume I focuses on the analysis of real-valued functions of a real variable. Besides developing the basic theory it describes many applications, including a chapter on Fourier series. It also includes a Prologue in which the author introduces the axioms of set theory and uses them to construct the real number system. Volume II goes on to consider metric and topological spaces, and functions of several variables. Volume III covers complex analysis and the theory of measure and integration.

D. J. H. GARLING is Emeritus Reader in Mathematical Analysis at the University of Cambridge and Fellow of St John's College, Cambridge. He has fifty years' experience of teaching undergraduate students in most areas of pure mathematics, but particularly in analysis.

Introduction

This book is the first of three volumes of a full and detailed account of those elements of real and complex analysis that mathematical undergraduates may expect to meet in the first two years or so of the study of analysis. This volume is concerned with the analysis of real-valued functions of a real variable. Volume II considers metric and topological spaces, and functions of several variables, while Volume III is concerned with complex analysis, and with the theory of measure and integration.

Mathematical analysis depends in a fundamental way on the properties of the real numbers, and indeed much of analysis consists of working out their consequences. It is therefore essential to develop a full understanding of these properties. There are two ways of doing this. The traditional and appropriate way is to take the fundamental properties of the real numbers as axioms – the real numbers form an ordered field in which every non-empty subset which has an upper bound has a least upper bound – and to develop the theory – convergence, continuity, differentiation and integration – from these axioms. This programme is carried out in Part Two. This theory is meant to be used, and Part Two ends with an extensive collection of applications. **The reader is strongly recommended to follow this tradition, and to begin at the beginning of Part Two.**

It is however right to ask about the foundations on which these axioms, and the rest of mathematical analysis, are built. These foundations are considered in the Prologue. In the twentieth century, analysis was placed in a set-theoretic setting, and it is worth understanding what this involves. Chapter 1 contains an account of Zermelo–Fraenkel set theory, together with a brief discussion of the axiom of choice and its variants. The Zermelo–Fraenkel axioms lead naturally to the construction of the natural numbers. In Chapter 2 it is shown that there is then a steady progression through the integers and the rational numbers to the real numbers and the complex numbers. The problem with the natural numbers, the integers and the rational numbers is that they are very familiar; this part of the journey may appear to be spent proving the obvious. The construction of the real numbers is a quite different matter. There are many possible constructions, but we describe the first, given by Richard Dedekind. This has great virtue, since it involves both order and metric properties of the rational numbers and of the real numbers. **The reader is urged to defer a detailed reading of the Prologue until the occasion demands,** for example when it becomes clear how important the fundamental properties of the real numbers are, or when it is important to consider carefully the role of induction, recursion and the axiom of dependent choice.

The text includes plenty of exercises. Some are straightforward, some are searching, and some contain results needed later. Many concern applications, and all help develop an understanding of the theory: do them!

I have worked hard to remove errors, but undoubtedly some remain. Corrections and further comments can be found on a web page on my personal home page at `www.dpmms.cam.ac.uk`.

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Part One

Prologue: The foundations of analysis