DHIREN KUMAR BASNET

TOPICS IN INTUITIONISTIC FUZZY ALGEBRA

An Introduction

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TOPICS IN INTUITIONISTIC FUZZY ALGEBRA

Dedicated to
my son

Babu

Preface.

Since the inception of the notion of a fuzzy set as a generalization of a crisp set in 1965 by L. A. Zadeh, which laid the foundation of fuzzy set theory, the literature of fuzzy set theory and its application has been growing rapidly, amounting by now to several thousands of papers. These are widely scattered over all the other disciplines such as economics, psychology, artificial intelligence, network analysis, decision making etc. The notion of Intuitionistic fuzzy set was introduced by K. T. Atanassov in 1986 as a generalization of fuzzy set and after the introduction of algebra in fuzzy setting by Rosenfeld in 1971, the research took a new direction called fuzzy algebra. In a similar way Intuitionistic fuzzy algebra was launched. This is a book form of the research done by the author during his doctoral degree work. The motivation for publishing it as a book is to help the researchers working in this field. Hope this book will provide necessary prerequisites for the beginners in this field.

Chapter 0 consists of some historical background of the subject, chapter 1 consists of some definitions and results used throughout the book. Chapters 2 and 3 consist of ideas of Intuitionistic fuzzy ideals of a ring, (α,β)-cut of Intuitionistic fuzzy ideals, relation between the images of Intuitionistic fuzzy ideals and Noetherian and Artinian rings. Chapters 4 and 5 contain ideas of Intuitionistic fuzzy essential submodules, Intuitionistic fuzzy uniform submodules, Intuitionistic fuzzy submodules of IF Goldie dimension etc. Finally chapter 7 contains the concepts of Intuitionistic fuzzy projective and injective submodules.

I offer my sincere regards to my supervisor Prof. B. Banerjee for his guidance and constant support during my research work. I also convey my heartfelt gratitutde to my parents, wife, son and other family members for everything they have done for me and without whose continuous encouragement this work could never be published. I also thank my colleagues in D.H.S.K. College, Assam, India and in Assam University, Silchar, Assam, India, my friends, students and well wishers for their encouragement. Thanks are due to my colleague and friend Naba Kanta Sarma and my student Krishnendu Das for pointing out mistakes and suggesting better presentation of the material. Last but not the least, thanks are also due to the publisher for his cooperation in bringing out this book.

D. K. Basnet.

LIST OF ABBREVIATIONS AND SYMBOLS

IFS	Intuitionistic fuzzy subset.
IFR	Intuitionistic fuzzy subring.
IFI	Intuitionistic fuzzy ideal.
IFM	Intuitionistic fuzzy submodule.
IFUM	Intuitionistic fuzzy uniform submodule.
Z	The set of integers.
R	The set of real numbers.
$\mathbf{Z}_n$	The ring (or module) of integers modulo n.
$< n >$ or n**Z**	The ideal (or submodule) generated by n.
$A \leq B$	A is a submodule of B (or A is an IFM of B).
$A \leq_e B$	A is an essential submodule of B.
$A \leq_{ef} B$	A is an IF essential submodule of B.
$A \oplus B$	Direct sum (Internal) of A and B.
$\prod_{i \in I} A_i$	Direct product of modules A_i's (or IFM A_i's).
$\sum_{i \in I} A_i$	Direct sum (External) of modules A_i's (or IFM A_j's).
A^*	A subset of M defined by $\{x \in M : \mu_A(x) > 0 \text{ and } \nu_A(x) < 1\}$.
$A\|_B$	Restricted IFS on B.

CONTENTS

CHAPTER 0

INTRODUCTION

0.1. *Historical Development.*

The process and progress of knowledge unfolds into two stages : an attempt to know the character of the world and a subsequent attempt to know the character of knowledge itself. Now a days we want to know not only specific facts but also what we can and cannot know, what we do and do not know and how we know at all. As we become aware of how much we know and of how much we do not know, as information and uncertainty themselves become the focus of our concern, we begin to see our problem as centering around the issue of complexity.

Many of the collections and categories often we encounter with, such as the class of tall people, expensive cars, highly contagious diseases, numbers much greater than 1 or sunny days do not have precisely defined criteria of membership. Instead their boundaries seem vague and the transition from member to nonmember appears gradual rather than abrupt.

To eliminate the sharp boundary dividing members of the class from the nonmembers, L. A. Zadeh [87] introduced the notion of weighted membership. An element may then belong more or less to a subset, and from there, Zadeh [87] introduced a fundamental concept, that of fuzzy set. The notion of fuzzy set provides a convenient point of departure for the construction of a conceptual framework which parallels in many respect the framework used in the case of ordinary sets, but is more general than the later and potentially may prove to have a much wider scope of applicability, particularly in the fields of pattern classification and information processing. Essentially such a framework provides a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership rather than the presence of random variables.

Since the inception of the notion of a fuzzy set in 1965 by Zadeh [87], which laid the foundation of fuzzy set theory, the literature on fuzzy set theory and its application has been growing rapidly amounting by now to several thousands of papers. These are widely scattered over many disciplines such as economics, psychology, network analysis etc.

0.2. *Earlier works done in this field.*

In 1971, Rosenfeld [80] used fuzzy set in the realm of group theory and formulated the concept of fuzzy subgroup of a group with respect to t-norm minimum. Since then, a host of mathematicians have been engrossed in extending the concepts and results of abstract algebra to boarder framework of fuzzy setting.

An exhaustive analysis of the subject from different points of view has been carried out and the definition of fuzzy subgroup given by Rosenfeld [80] in 1971 was the first step in this direction. Rosenfeld [80] confined his attempt into more basic concepts of fuzzy algebra following the notion and terminology as introduced by Zadeh [87].

In 1979 Anthony and Sherwood [4] redefined fuzzy subgroup, Das [30] introduced the idea of level subsets, which gave a new dimension to the fuzzy set theory and as a result Mukherjee and Bhattacharya [18] showed that almost all the global notions of fuzzy subgroup can be characterized by its level subgroups. Sherwood [81] also introduced the notion of product of fuzzy subgroups and proved some interesting results on product of fuzzy subgroups.

In fuzzy group theory and in other branches of fuzzy algebraic structures, by a homomorphism we always mean a classical homomorphism from an algebraic structure of one variety to another algebraic structure of the same variety. Using this classical homomorphism, images and pre-images of fuzzy sets are defined and their properties are investigated. Choudhury, Chakraborty and Khare [29] introduced the notion of fuzzy homomorphism between two groups and studied its effect on fuzzy subgroups.

Chakraborty and Khare [24] also studied the composition of fuzzy homomorphisms and proved an analogue of the fundamental theorem of homomorphism of groups.

Ajmal [1] defined the notion of containment of an ordinary kernel of a group homomorphism in fuzzy subgroup. Using this idea, he provided the solution of the problem of showing a one-to-one correspondence between the family of fuzzy subgroups of a group containing the kernel of a given homomorphism, and the family of fuzzy subgroups of the homomorphic image of the given group. It was shown that an ordinary kernel gave rise to the notion of fuzzy quotient group in a natural way. Consequently, the fundamental theorem of homomorphism was established for fuzzy subgroups. Moreover, he provided new proofs for the facts that the homomorphic image of a fuzzy subgroup and fuzzy normality are invariant under epimorphism.

The idea of fuzzy subrings and ideals was introduced by Wang Jin-Liu [84] in 1982. Subsequently among others, Wang Jin-Liu himself [84] fuzzified certain standard results on rings and ideals.

N. Kuroki [62] studied fuzzy ideals and bi-ideals in semi-groups. Wang Jin-Liu [85] studied fuzzy ideals of a ring. Mukherjee and Sen [75, 76] discussed fuzzy prime ideals, fuzzy completely prime ideals, and fuzzy weakly completely prime ideals of a ring. Subsequently among others, Swamy and Swamy [82], Kumbhojkar and Bapat [49] extended various concepts and results of the ring theory to the fuzzy settings. Nanda [78] discussed fuzzy modules over fuzzy rings. A study of various kinds of fuzzy ideals was carried by Kumar [55].

The concept of fuzzy ideals was extended by Kumar [52] by introducing fuzzy semi-primary ideals in rings. This class of fuzzy ideals generalized the class of fuzzy semi-primary ideals. It was shown that there did not exist fuzzy ideals, which were fuzzy semi-primary but not fuzzy primary. After this, he [54] introduced the concept of a fuzzy nil radical of a fuzzy ideal in a ring. Using this concept, he defined fuzzy primary ideal and obtained some fundamental results concerning these notions.

Fuzzy prime ideals were further investigated by Malik and Mordeson [72]. They introduced the concept of fuzzy homomorphism of rings. They proved the results for fuzzy homomorphism analogous of the homomorphism theorems of ring homomorphism.

Kuraoka and Kuroki [61] defined fuzzy quotient ring induced by fuzzy ideals and studied the relation between fuzzy quotient rings and fuzzy ideals.

An internal description of the fuzzy subring and fuzzy ideal generated by a finite fuzzy subset of a ring was provided by Dixit, Kumar and Ajmal [33]. They defined the fuzzy coset of fuzzy ideal and for the set of all fuzzy cosets of a fuzzy ideal, a ring structure was given. A fuzzy semi-prime ideal was also defined and some basic properties of this ideal were studied.

In 1992, Kumar [57] again defined fuzzy semi-primary ideal by using the idea of the fuzzy nil radical of a fuzzy ideal. The level ideals of such a fuzzy ideal were examined and the ring of fuzzy cosets of such a fuzzy ideal was studied. The algebraic nature of such fuzzy ideals under homomorphism was also ascertained.

Kumbhojkar and Bapat [50] discussed the merits and demerits of various definitions of prime and primary fuzzy ideals, and new definitions were suggested. The nil radical was defined as against the (prime) radical of a fuzzy ideal. Some theorems regarding the nil radical were proved. It was shown that the two radicals coincide when the grade membership lattice iss totally ordered.

The concept of fuzzy set was extended by R. Kumar [53] with the introduction of the concept of fuzzy irreducible ideal in a ring. He also examined the nature of homomorphic images and pre-images of fuzzy irreducible ideals.

Dutta and Biswas [36] studied fuzzy k-ideals of semi-rings and discussed some properties. Chang Bum Kim and Mi-Ae Park [26] introduced k-fuzzy ideals of a semiring. They constructed an extension of a fuzzy ideal in a k-semi-ring and studied the quotient structure of a k-semiring by a fuzzy ideal.

Chang Bum Kim [27] studied fuzzy maximal k-ideal of a k-semiring. He characterized the quotient k-semiring of a k-semiring by a fuzzy maximal k-ideal and proved some isomorphism theorems in commutative k- semirings.

Malik and Mordeson [70] introduced the concepts of the fuzzy weak direct sum and the fuzzy complete direct sum of fuzzy subrings of commutative rings.

The chain rings were characterized by Alkhamees [3] in terms of fuzzy ideals by using the concept of minimal generating set. He also studied the concept of fuzzy direct sum to characterize Artinian principal ideal rings and established the necessary and sufficient condition under which an Artinian ring is a direct sum of fields.

In their classical paper 'rings with chain conditions' Mukherjee and Sen [77] characterized Artinian and Noetherian rings by fuzzy ideals and presented alternative proofs of some well known results of Artinian and Noetherian rings.

The notion of product of fuzzy ideals and fuzzy submodules was introduced by Kumar, Bhambri, Kumar [60]. They also defined the sum of two fuzzy submodules and discussed the fuzzy version of Nakayama's Lemma.

The notion of intuitionistic fuzzy set was introduced by Atanassov [8] as a generalization of Zadeh's fuzzy sets in 1986. In that paper, he also defined various operations on intuitionistic fuzzy sets and showed that these operations were different from the similar operations on fuzzy sets. He [9, 13] also proved some interesting theorems on intuitionistic fuzzy sets. R. Biswas [20] defined anti fuzzy subgroup and discussed some of its properties. In that paper, he studied the concept of lower level subset of a fuzzy set, lower level subgroup and proved some results. He [21] also introduced intuitionistic fuzzy subgroup and made some characterizations.

In the paper 'Structure on intuitionistic fuzzy relations', Bustince and Burillo [23] studied the structures of intuitionistic fuzzy relations and analised the existence relations between the structures of a relation and the structures of its complementary one. In [22] they defined the distance measure between intuitionistic fuzzy sets and gave an axiomatic definition of intuitionistic fuzzy entropy and their characterization.

Jun, Kim and Yon [44] applied the concept of intuitionistic fuzzy sets to ideals of a near ring, introduced the notion of an intuitionistic fuzzy ideal of a near ring and investigated some related properties. Also they characterized the intuitionistic fuzzy ideals of a near ring. In [86] they introduced the notion of intuitionistic fuzzy R-subgroups of a near ring and equivalence relations on the family of all intuitionistic fuzzy R-subgroups of a near ring.

Maji, Biswas and Roy [65] introduced the notions of intuitionistic fuzzy soft sets. Jun and Dudek [43] considered the intuitionistic fuzzyfication of subalgebras and closed ideals in BCH-algebras and established the relation between an intuitionistic fuzzy sub algebra and intuitionistic fuzzy closed ideals.

Qin, Qiao and Chen [79] discussed the properties of quotient lie algebra that was defined by fuzzy lie ideals. Also they discussed the images of fuzzy Lie algebra and fuzzy Lie ideal under generalized extension principle.

In [41] the idea of essential submodule, closed submodule, relative complement for a submodule was given by Goodearl and many interesting results involving them were proved. Also the notion of finite dimensionality of a module, uniform module, modules of Goldie dimension, was described.

Similarly Lambek [64] presented beautifully the notions of free module, projective and injective modules and established many results.

0.3. *Present Investigation.*

Chapter 0 consists of the general historical development of fuzzy and intuitionistic fuzzy sets and of different algebraic structures in fuzzy and intuitionistic fuzzy setting.

In chapter 1, we discuss some definitions and important results from ordinary algebra as well as from fuzzy algebra, which have been used throughout the book in order to make the book self-sufficient.

Chapter 2 deals with intuitionistic fuzzy subrings and intuitionistic fuzzy ideals, the sum and product of two intuitionistic fuzzy ideals, f-invariant intuitionistic fuzzy ideals. Also the images and inverse images of intuitionistic fuzzy ideals under homomorphism of rings are studied and it is proved that the sum and product of two intuitionistic fuzzy ideals of a ring is again an intuitionistic fuzzy ideal of the ring, according to our definition although that is not true for the definitions of sum and product of two intuitionistic fuzzy sets given by Atanassov [8]. Finally we established the correspondence theorem for intuitionistic fuzzy ideals.

Intuitionistic fuzzy ideals of quotient ring of a ring, finite and infinite images of intuitionistic fuzzy ideals are studied in the chapter 3 and some characterization of Artinian and Noetherian rings are given. Finally we presented alternative proofs of some well-known results of Artinian and Noetherian rings in terms of intuitionistic fuzzy ideals.

In chapter 4, we introduced the notion of intuitionistic fuzzy submodule, intuitionistic fuzzy essential submodule, intuitionistic fuzzy relative complement, intuitionistic fuzzy closed submodule and established many interesting relations among them. Also we defined the intuitionistic fuzzy quotient submodule of an intuitionistic fuzzy submodule and direct sum of intuitionistic fuzzy submodules.

In chapter 5, the ideas of finite dimensional intuitionistic fuzzy submodule, intuitionistic fuzzy uniform submodule and intuitionistic fuzzy submodule of IF-Goldie dimension are introduced and some interesting results pertaining to these are presented.

Finally in chapter 6, we introduced the notion of free intuitionistic fuzzy submodule, intuitionistic fuzzy projective and injective submodules. First we have introduced the concept of homomorphism between two intuitionistic fuzzy submodules, direct product and direct sum of intuitionistic fuzzy submodules and then some important results are proved.

* * *

CHAPTER 1

PREREQUISITES

In this chapter we discuss some definitions and important results from algebra as well as from fuzzy algebra, which have been used throughout the book in order to make the book self sufficient.

1.1. Definition. [87] A fuzzy subset μ of a set E is a function $\mu : E \to [0, 1]$. The function μ is called membership function.

1.2. Definition. [8] Let E be any set. An intuitionistic fuzzy subset (in short IFS) A^* of E is an object of the following form $A^* = \{< x, \mu_A(x), \nu_A(x) > | x \in E\}$ where $\mu_A: E \to [0, 1]$ and $\nu_A : E \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in E$ respectively and for every $x \in E$, $0 \le \mu_A(x) + \nu_A(x) \le 1$. For simplicity we shall write A instead of A^*.

1.3. Definition. [8] If $A = \{< x, \mu_A(x), \nu_A(x) >| x \in E\}$ and $B = \{< x, \mu_B(x), \nu_B(x) >| x \in E\}$ be any two IFS of a set E then

$A \subseteq B$ if and only if for all $x \in E$, $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$

$A = B$ if and only if for all $x \in E$, $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$

$\overline{A} = \{< x, \nu_A(x), \mu_A(x) >| x \in E\}$

$A^c = \{< x, \mu_A^c(x), \nu_A^c(x) >| x \in E\}$, where $\mu_A^c(x) = 1 - \mu_A(x)$ and $\nu_A^c(x) = 1 - \nu_A(x)$

$A \cap B = \{< x, (\mu_A \cap \mu_B)(x), (\nu_A \cup \nu_B)(x) >| x \in E\}$, where

$(\mu_A \cap \mu_B)(x) = \min\{\mu_A(x), \mu_B(x)\}$ and $(\nu_A \cup \nu_B)(x) = \max\{\nu_A(x), \nu_B(x)\}$

$A \cup B = \{< x, (\mu_A \cup \mu_B)(x), (\nu_A \cap \nu_B)(x) >| x \in E\}$, where

$(\mu_A \cup \mu_B)(x) = \max\{\mu_A(x), \mu_B(x)\}$ and $(\nu_A \cap \nu_B)(x) = \min\{\nu_A(x), \nu_B(x)\}$

$\Box A = \{< x, \mu_A(x), 1 - \mu_A(x) > | x \in E\}$, $\Diamond A = \{< x, 1 - \nu_A(x), \nu_A(x) > | x \in E\}$.

Also a fuzzy set has the form $\{< x, \mu_A(x), \mu^c_A(x) >| x \in E\}$.

1.4. Definition. [21] Let G be a group and $A = \{ < x, \mu_A(x), \nu_A(x) > | x \in G\}$ be an IFS of G. Then A is said to be an intuitionistic fuzzy subgroup (in short IFG) of G if the followings are satisfied :

(i) $\mu_A(xy^{-1}) \ge \min\{\mu_A(x), \mu_A(y)\}$

(ii) $\nu_A(xy^{-1}) \le \max\{\nu_A(x), \nu_A(y)\}$ for all $x, y \in G$

1.5. Definition. [64] A commutative ring R is said to be a Noetherian ring, if for every ascending chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ of ideals of R, there exists a positive integer n such that $I_m = I_n$ for all $m \ge n$.

1.6. Definition. [64] A commutative ring R is said to be an Artinian ring if for every descending chain $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$ of ideals of R, there exists a positive integer n such that $I_m = I_n$ for all $m \ge n$.

1.7. Theorem. [64] If $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ is an infinite ascending chain of ideals of R, then $\cup\{A_i \mid i \in I\}$ is also an ideal of R.

1.8. Definition. [64] Let M be an additive abelian group and R be a ring. Then M is said to be a left module over R or a left R-module if there exists a mapping $R \times M \to M$, denoted by juxtaposition, such that :

$r(m + n) = rm + rn$,

$(r + s)m = rm + sm$,

$(rs)m = r(sm)$ for all $r, s \in R$; $m, n \in M$.

Similarly we can define right R-module. A left R-module is said to be a unitary module if $1m = m$ for all $m \in M$, where 1 is the unity element of R.

1.9. Definition. [64] Let M and M′ be two left modules over the same ring R, a mapping $f : M \to M'$ is said to be a module homomorphism if

$f(m + n) = f(m) + f(n)$ and $f(rm) = rf(m)$ for all $r \in R$ and $m, n \in M$.

1.10. Definition. [64] A finite family $\{A_1, A_2, \ldots, A_n\}$ of nonzero submodules of a module M is said to be independent if $A_i \cap (A_1 + A_2 + \ldots + A_{i-1} + A_{i+1} + \ldots + A_n) = \{0\}$ for all $i = 1, 2, 3, \ldots, n$. An infinite family $\{A_i\}$ of nonzero submodules of M is said to be independent if every finite sub family of $\{A_i\}$ is independent.

1.11. Definition. [64] If a family of submodules $\{A_1, A_2, \ldots, A_n\}$ of a module M is independent then their sum $A_1 + A_2 + \ldots + A_n$ is called the direct sum of $A_1, A_2, \ldots, A_n$ and it is denoted by $A_1 \oplus A_2 \oplus \ldots \oplus A_n$. Also every element of the direct sum is uniquely expressible as $a_1 + a_2 + \ldots + a_n$, where $a_i \in A_i$ for $i = 1, 2, 3, \ldots, n$.

1.12. Definition. [41] A module M over a ring is said to be finite dimensional if it has no infinite independent family of nonzero submodules.

1.13. Definition. [64] The direct product $M = \prod_{i \in I} M_i$ of a family of modules M_i over a ring R, is their cartesian product with operations defined componentwise. Thus if $m \in M$ then $m : I \to \cup\{M_i \mid i \in I\}$ with $m(i) \in M_i$ for all $i \in I$. Clearly M is also a module over R.

The (external) direct sum $\sum_{i \in I} M_i \subseteq \prod_{i \in I} M_i$ of a family of modules M_i over a ring R consists of all

$m \in \prod_{i \in I} M_i$ for which $m(i) = 0$ for all but finitely many $i \in I$

1.14. Definition. [64] If $M = \prod_{i \in I} M_i$,then the canonical epimorphism $p_i : M \to M_i$ and canonical monomorphism $k_i : M_i \to M$ are defined by

$$p_i(m) = m(i)$$

and

$$k_i(m_i)(j) = \begin{cases} m_i & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases}$$

Clearly $p_i \circ k_i = I_i$, the identity mapping on M_i.

1.15. Definition. [41] Let A and B be two submodules of a module and $A \subseteq B$, then A is said be an essential submodule of B if every nonzero submodule of B has nonzero intersection with A and we denote it by $A \leq_e B$. In this case we say that B is an essential extension of A.

1.16. Definition. [41] Let A be a submodule of C. A relative complement for A is a submodule B of C which is maximal with respect to the property that $A \cap B = \{0\}$.

1.17. Definition. [41] A submodule A of a module C is said to be a closed submodule of C if A has no proper essential extensions inside C, i.e. if the only solution of the relation $A \leq_e B \leq C$ is $B = A$.

1.18. Proposition. [41] $\{0\}$ is an essential submodule of a module A only if $A = \{0\}$.

1.19. Proposition. [41] (a) If $A \leq B \leq C$ then $A \leq_e C$ if and only if $A \leq_e B \leq_e C$.

(b) If $A \leq_e B \leq C$ and $A' \leq_e B' \leq C$, then $A \cap A' \leq_e B \cap B'$.

(c) If $f : B \to C$ is a homomorphism and $A \leq_e C$ then $f^{-1}(A) \leq_e B$.