Introduction
to
Global Analysis

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Preface

The classical tradition of mathematical analysis studies functions, vector fields, differential equations, etc., in Euclidean space $\mathbb{R}^n$. It has become apparent in recent years that there is great interest in the methods of mathematical analysis carried out on manifolds, that is, spaces that look like $\mathbb{R}^n$ in the small. They have found concrete applications to the fields of physics, engineering, and economics; possible applications are under investigation in many other areas.

The present book aims at making much of this material available to anyone who is at the level of a beginning graduate student in mathematics. The first half of the book requires little more than a good foundation in undergraduate mathematics. In later chapters we use some integration, Fourier analysis, and homology theory. At appropriate places, we outline all the theory that we need, and we list many references. In the Introduction we sketch the history of the subject and give a detailed description of the individual chapters. A full list of references is given at the end of the book.

The material presented in this book was first exposed by the author in a graduate course at the University of Minnesota in 1976–1977. The course was very successful, attracting both students and faculty from mathematics and the sciences. I want to thank my listeners and friends for helping me to fix my ideas on the subjects presented here. I am also grateful for the remarkable typing talents of Peggy Gendron. As usual, responsibility for accuracy, interest, etc., lies entirely with me.

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In the *Batelle Rencontres* [8], a collection of papers dating from 1967, based on lectures in mathematics and physics, there is a substantial amount of what we might now call global analysis. More recently, the field has been shown to be of interest in engineering; for example, the survey article of Brockett [14] for the Institute of Electrical and Electronics Engineers is deeply involved with manifolds, vector fields, etc. I think, however, that almost everyone would agree that it is the catastrophe theory of Thom (e.g., [13, 108]) that has most caught the imagination of the general public. Surely some of this theory is controversial, and some of its more flamboyant applications will have to be reformulated or restrained to conform with the accepted rigor of science, but I doubt if anyone will deny that there is a certain amount of new mathematics, with a significant potential for application.

Where, then, is the middle ground of global analysis, and why would I want to risk writing a book in this young field? The foundation of any rational view of this field must be the study of smooth manifolds and the maps between them. Because I want to make this subject accessible to many people and am not afraid to do something once and then generalize it later (when there is a clear advantage in good exposition), I begin this text with a basic discussion of (finite-dimensional) differentiable manifolds. The first three chapters all are concerned with that area. I begin with those aspects of this theory that flow out of the usual advanced calculus, and I carry it through to prove versions of the Whitney embedding theorem, the theorem of Sard on the measure of the set of critical values, and finally, the transversality lemma of Thom.

With the foundations set, we proceed to look at the tangent bundle to a manifold, and more generally the theory of vector bundles (which I treat fully in the real case over a compact space). I feel that a certain amount of bundle theory is essential here, but that it would be a perversion of the general purpose of this text to include a full study of fiber bundles. For this reason I limit myself as indicated, rather in the spirit of the beautiful book of Atiyah [4].

This leads up to what may be considered the first topic belonging properly to global analysis, the general study of differential operators on manifolds. This text being an introduction, my goal here is to bring the student to where he or she may begin to learn the Atiyah–Singer index theorem. I make no attempt to offer any competition to the existing excellent, but rather more advanced, expositions in that area. It goes without saying that by that point, I must cover the basic topics of differentiation and integration on manifolds, including of course the algebra of differential forms, Stokes' theorem, the Poincaré lemma, and the basic definition of deRham cohomology. But no material from algebraic topology is presumed to be known by the reader up to this place in the book.

It is my belief that this is the proper place to extend some of our earlier material on differentiable manifolds to the infinite-dimensional case. To say something meaningful here would require a certain knowledge of functional analysis. We include some short references to the relevant parts of that field, hoping to guide the reader to appropriate places in the literature. Much of the earlier material about manifolds, smooth maps, tangent bundles, etc. carries over without serious difficulty to the more general case of infinite-dimensional manifolds. We do prove one basic result, which exhibits the function space of maps between manifolds as an infinite-dimensional manifold. We also discuss, and prove part of, Kuiper's well-known result concerning the contractability of the infinite-dimensional linear group. But to keep within the general scope I have in mind here, I must stop short of more recent developments, such as the work of Kuiper, Eells, Elworthy, and others on Hilbert manifolds.

The remainder of this text consists of a series of four topics, all closely related to analysis on manifolds:

(i) Morse theory, the study of smooth functions at critical points;
(ii) Lie groups and their actions on manifolds;
(iii) dynamical systems and structural stability;
(iv) a descriptive introduction to singularities and catastrophies.

We aim here for a solid description of the basic results in these areas. For example, one needs some basic facts from topology (rational homology groups, Euler characteristic, etc.) to discuss the Morse inequalities. We include an outline of this material, along with many references, at the appropriate point. In the case of topic (iv) we aim only to describe the general ideas; we mention the seven elementary catastrophies, but do not prove that they are exhaustive in the dimensions in question.

No author would be entirely honest if he did not clearly indicate what relevant material has been deliberately omitted from the book. First of all, we have stressed the compact case. Much of what we have said about manifolds, vector bundles, etc. is not, in fact, limited by that restriction. Unfortunately, the methods of proof in the general cases are frequently much more involved. Second, we never enter properly into the domain of K-theory, a topic that is both beautiful and naturally related to the material of this text. It is unfortunate that *K*-theory presumes much more algebraic topology than I felt could be safely assumed as a prerequisite for this text. Third, for many of the later topics in the book, I strive to give a clear idea of what is involved in the area rather than give maximal known generality. Examples of this are the sections on infinite-dimensional manifolds, the discussion of Lie groups (minimal reference to Lie algebras), and the frequent restriction to the real case (with at most a remark about the equally important complex case). Of course I have tried to give full lists of references, so that a serious reader will not have too much trouble tracking things down.

The original text was intended to be available to a well-prepared first-year graduate student. I had to assume a full foundation in basic analysis (still often taught under the misleading name of "advanced calculus"), linear algebra, and point-set topology. As I indicated above, I have had to stray from this at a few points, but I think a serious and patient student can survive these hurdles. I do not think that the basic material from algebraic topology (as in the chapter on Morse theory) or the ocassional reference to Lebesgue measure, initially restricted to a discussion of sets of Lebesgue measure zero, need be regarded as a major obstacle. Most students who are at the level of this book are simultaneously learning some of these important topics. On occassion, some of these additional topics may be taken on faith for a short period of time to accommodate the contingencies of one's program. In a similar way, I have felt free to use the basic existence and uniqueness theorems from differential equations.

I have decided against a formal list of problems at the end of every section. I have specifically indicated where good sources of relevant problems may be found, and have otherwise indicated some projects relevant to the material at hand. At some places, however, I do state specific problems. Naturally, working some problems is essential in getting an understanding of the subject.

On the other side of the coin, mature mathematicians may well find some interest in portions of this book. For them the basic material will probably be boring, and it will be necessary to feel their way around to find where new and interesting material first occurs. I hope that nobody will be angry; it is a lot easier to skip pages than to fill in material that one does not know. I will personally feel that I have succeeded if I can bring students, as well as faculty whose interests lie elsewhere, to the point of understanding colloquium lectures in this beautiful field.

CHAPTER

1

Manifolds and Their Maps

The purpose of this chapter is to lay the groundwork for the study of smooth manifolds and their maps. We begin with their general properties and then review quickly the important special case of Euclidean spaces. No political treaty has ever been drawn-up to delineate clearly between advanced calculus and the elementary theory of manifolds. I shall attempt, toward the end of this chapter, to give a concise presentation of this borderline material, hoping to be helpful to some without offending others.

DIFFERENTIABLE MANIFOLDS AND THEIR MAPS

A manifold is a nice topological space that in the small is just like Euclidean space (a rigorous definition will follow). To do mathematical analysis on manifolds, one imposes some condition of smoothness or differentiability concerning the relation of any two nearby pieces, each of which is like Euclidean space. To assure that the thing does not get too large, one usually imposes some mild set-theoretic conditions. A typical example would be the ordinary sphere (the surface of the earth). A nice small region—such as a polar ice cap—is virtually indistinguishable from a small region in the plane, while the entire manifold, the sphere, is fundamentally different from the plane. We shall define a manifold first, and after some examples turn to differentiable manifolds.

Definition 1.1 An n-manifold (or manifold of dimension n) is a topological space M^n that satisfies the following:

(a) M^n is a Hausdorff space (satisfying the separation axiom T_2 of Hausdorff).

(b) If x is a point of M^n, then there is an open set $O \subseteq M^n$, $x \in O$, with O homeomorphic to the Euclidean space $\mathbb{R}^n$ ($\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$, n factors).

(c) M^n has a countable basis for its topology; i.e., there is a countable family of open sets $\{O_m\}$ such that every open set is a union of some of the O_m's.

Examples *1.* The Euclidean spaces $\mathbb{R}^n$ are trivial examples (take a single open set $O = \mathbb{R}^n$).

2. Any open set $U \subseteq \mathbb{R}^n$ is clearly an n-dimensional manifold because, as is easily seen, an open ball of fixed radius about a point in $\mathbb{R}^n$ is homeomorphic to all of $\mathbb{R}^n$. (Using polar coordinates, the required homeomorphism is the identity on all angular coordinates and on the radial coordinate can be chosen to be any homeomorphism from the interval in question to the entire real numbers, and is easily constructed from functions such as the tangent.)

More generally, if M^n is an n-manifold and $U \subseteq M^n$ is an open subset, then U is an n-manifold.

3. Let $S^n = \{X \in \mathbb{R}^{n+1} | d(\mathbb{O}, X) = 1\}$, where $d(\mathbb{O}, X)$ means the distance from X to the origin $\mathbb{O}$. This is the standard n-dimensional sphere. If $X = (x_1, \ldots, x_{n+1})$, then we set

$$U = \{X \in S^n | x_{n+1} > -1\}, \qquad V = \{X \in S^n | x_{n+1} < +1\}.$$

Clearly U and V are a covering of S^n by two open sets.

Now, if $X \in U$, let l_X denote the straight line in $\mathbb{R}^{n+1}$ through the two points $(0, 0, \ldots, -1)$ and X. Then clearly l_X meets the hyperplane

$$H_1 = \{X \in \mathbb{R}^{n+1} | x_{n+1} = 1\}$$

at a single point, called $\phi(X)$. Then ϕ is trivially checked to be a homeomorphism from U to H_1. But the projection onto the first n coordinates yields a homeomorphism between H_1 and $\mathbb{R}^n$. Thus U is homeomorphic to $\mathbb{R}^n$.

Similarly, V is homeomorphic to $\mathbb{R}^n$. S^n is obviously a compact Hausdorff space with a countable basis; thus it is an n-manifold

4. If M^n and N^k are manifolds of dimension n and k, respectively, then $M^n \times N^k$ is easily seen to be a manifold of dimension $n + k$. The torus $S^1 \times S^1$ is an easy example.

5. Let $\mathrm{Gl}(n; \mathbb{R})$ be the group of $n \times n$ matrices over the real numbers with nonvanishing determinant. It is a topological space because it is a subspace of all $n \times n$ matrices that is topologized exactly as Euclidean space of dimension n^2. Alternatively, we may define the norm of a matrix (a_{ij}),

$$\|(a_{ij})\| = \sqrt{\sum_{i,j} (a_{ij})^2},$$

and then a metric by

$$d((a_{ij}), (b_{ij})) = \|(a_{ij} - b_{ij})\|.$$

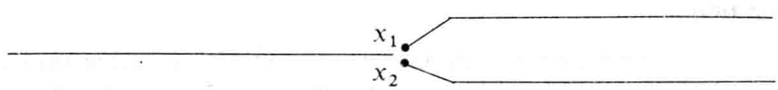
If we set

$$D = \{(a_{ij}) \mid \det(a_{ij}) = 0\},$$

then $\mathrm{Gl}(n; \mathbb{R})$ is clearly the complement of D in the set of all $n \times n$ matrices. But D is trivially checked to be closed, so $\mathrm{Gl}(n, \mathbb{R})$ is open and hence a manifold of dimension n^2.

This should suffice to show that the collection of manifolds is very substantial; we shall meet many other examples as we proceed through this book. Before turning to differentiable manifolds, it is important that we look for a moment at a few technical points connected with the definition of a manifold.

(1) All of the hypotheses made in the definition of an n-manifold are independent of one another. For example, condition (b) in Definition 1.1 does not imply that M^n is Hausdorff. Consider the following space in the form of a letter Y



where the left branch is an open interval and the two right branches are half open and contain their left-hand endpoints. The topology around every point other than x_1 and x_2 is the usual topology of the real line. An open neighborhood of x_1 consists of the union of a half-open interval in the upper-right segment with x_1 as leftmost point and an open subset of the left branch consisting of all points in that branch that are to the right of a given point. Similarly, an open neighborhood of x_2 is defined using points of the lower-right segment. In particular, any open neighborhood of either x_1 or x_2 must contain all points to the right of some given point in the left-hand branch.

This topological space has the property that every point has an open neighborhood homeomorphic to $\mathbb{R}^1$, but the two points x_1 and x_2 do not have disjoint open neighborhoods, so that it is not a Hausdorff space.

There are classic examples of "manifolds" for which condition (c) fails, the so-called long line, for example (see [5]).

(2) An n-manifold in the sense of Definition 1.1 is ocasionally called a topological manifold, to distinguish it, for example, from a differentiable manifold (Definition 1.2).

(3) An n-manifold M^n enjoys various point-set theoretic properties in a somewhat automatic fashion. For example,

(i) A manifold M^n is locally compact. If $x \in M^n$, choose an open neighborhood O of x that is homeomorphic to $\mathbb{R}^n$. Around the point x in O we may then select an open ball of finite radius (regarding O as $\mathbb{R}^n$). The closure of such an open ball is, of course, compact.

(ii) M^n is separable (that is, it has a countable dense subset). This follows at once from Definition 1.1c.

(iii) M^n is regular; that is, if x and C are a point and a closed subset of M^n with $x \notin C$, then x and C are contained in disjoint open sets. For this it suffices to handle the case where both x and C are contained in an open neighborhood of x homeomorphic to $\mathbb{R}^n$. This is trivial, for $\mathbb{R}^n$ is a metric space.

(iv) With slightly more work, M^n may be shown to be a normal space.

(v) Combining Definition 1.1c with (iv) or (iii) above and using the Urysohn metrization theorem, we see that an n-manifold in the sense of Definition 1.1 is always a metric space. This will be greatly strengthened in the case of a smooth manifold when we construct a Riemann metric. Furthermore, we shall prove, in the case of a compact smooth manifold, that M^n is always homeomorphic to a subspace of $\mathbb{R}^{2n+1}$. This embedding theorem, which may also be established without compactness, trivially implies metrization.

(4) The number n, the dimension of an n-manifold, is in fact an invariant. That is, if two manifolds are homeomorphic, then they have the same dimension. This result is significantly deeper than one might guess on first looking at it. Even when the manifolds are restricted to be simple Euclidean spaces, it is nontrivial. The assertion that $\mathbb{R}^n$ and $\mathbb{R}^m$ can only be homeomorphic when $n = m$ is the famous theorem on invariance of dimension from algebraic topology. The general fact that the dimension of a manifold is an invariant follows by similar methods of algebraic topology (see, e.g., [25, 33]).

In this context, we might also mention another relevant theorem from algebraic topology, the invariance of domain. This asserts that if U_1 and U_2 are homeomorphic subsets of a given manifold M^n, and U_1 is open in M^n, then so is U_2. (Note that the homeomorphism between these subsets is not assumed to extend to a homeomorphism of the entire manifold to itself, in which case the theorem would, of course, be trivial.)

On the other hand, for differentiable manifolds (soon to be defined) the fact that dimension is an invariant is rather trivial, and we shall prove it later in this chapter (Corollary 1.5).

(5) Naturally, a manifold need not be connected. For example, $\mathrm{Gl}(n, \mathbb{R})$ has two connected components, corresponding to the positive and the negative determinants. But if a manifold is connected, it is automatically pathwise connected. For let x_0 be a given point in the manifold M^n and let M_0 be the set of points that may be connected to x_0 by a continuous path. Using the fact that the manifold is locally like Euclidean space, it is easy to see that M_0 is both open and closed. Since M^n is connected, we must have $M_0 = M^n$, proving that M^n is pathwise connected.

Our principal interest in this text will be manifolds with some differentiability or smoothness. This can be viewed in two ways. First, any point lies in an open set that is homeomorphic to $\mathbb{R}^n$. If more than one such open set overlap or mesh badly, the change of coordinates obtained by viewing a given point in two different open sets might be a bad sort of function. But if the open cover by sets homeomorphic to $\mathbb{R}^n$ can be chosen nicely, then the possible change of coordinates arising from viewing a given point as lying in two overlapping open sets might always be a differentiable function from a subset of $\mathbb{R}^n$ to another. This is the first (and most common) way of getting at the notion of a differentiable manifold.

A second possible approach is in terms of functions. Given a function $f: M^n \to \mathbb{R}$ and a point $x \in M^n$ contained in an open set O that is homeomorphic to $\mathbb{R}^n$, we might wish to call f differentiable at x, provided f restricted to O is differentiable at x, in terms of the usual notion of differentiability of functions defined on Euclidean space. Unfortunately, this will not be an intrinsic property of the function f and the point x, but will depend on the choice of open set O. If there is a cover by suitable open sets, each homeomorphic to $\mathbb{R}^n$, such that the question of whether a function is differentiable at any given point x has a consistent answer, not dependent on the choice of a particular open set, then the manifold clearly ought to be thought of as differentiable. This is an alternative and equivalent approach, first exposed in [20], that we shall also sketch.

Here then is the official definition.

Definition 1.2 A manifold is called smooth (or infinitely differentiable, or sometimes just differentiable) if there is a family of open sets $\{O_\alpha\}$ that cover the manifold, each of which is homeomorphic to $\mathbb{R}^n$ by a given homeomorphism

$$\phi_\alpha: O_\alpha \to \mathbb{R}^n$$

such that, whenever $O_\alpha \cap O_\beta$ is nonempty, the composite map

$$\phi_\beta(O_\alpha \cap O_\beta) \xrightarrow{(\phi_\beta | O_\alpha \cap O_\beta)^{-1}} O_\alpha \cap O_\beta \xrightarrow{\phi_\alpha | O_\alpha \cap O_\beta} \mathbb{R}^n,$$

whose domain is visibly an open subset of $\mathbb{R}^n$, has continuous partial derivatives of all orders.

The pair (O_α, ϕ_α) is often called a chart or a coordinate chart.

Remarks (1) Our previous examples furnish many examples of differentiable or smooth manifolds. $\mathbb{R}^n$ and all its open subsets are obviously smooth manifolds. One may easily check that S^n, with the cover by two open sets given above, is a smooth manifold.

(2) Obviously, a smooth or differentiable manifold is a manifold, but it is remarkable that the reverse assertion is false. Kervaire [63], and later Smale and others, have shown that a compact manifold need not be differentiable. Kervaire constructs a 10-dimensional manifold and shows that it is not possible to select an open cover by sets meeting the condition of Definition 1.2. (Actually, the proof is rather indirect and depends on considerable algebraic topology.)

(3) Given a topological manifold, Remark (2) shows that there is an existence problem, the question whether one can find a differentiable structure on that manifold. Precisely, one may ask whether there is a smooth manifold that is homeomorphic to the original manifold. Similarly, there is a uniqueness problem, which we shall formulate precisely below. In a very famous paper, Milnor [76] has shown that the 7-dimensional sphere has a plurality of differentiable structures.

(4) Occasionally one needs to study various degrees of differentiability. If we require that the maps in Definition 1.2 have continuous partial derivatives of all orders less than or equal to some nonnegative integer k, then we say that the manifold is C^k, or differentiable of class k. In this terminology a smooth manifold is a C^∞ manifold, while an ordinary or topological manifold (Definition 1.1) is a C^0 manifold. A key theorem of Whitney [116] asserts that if a manifold has a C^k differentiable structure—that is, a cover by coordinate charts such that the maps in Definition 1.2, $(\phi_\alpha | O_\alpha \cap O_\beta) \circ (\phi_\beta | O_\alpha \cap O_\beta)^{-1}$, have continuous partials of order less than or equal to k (with $k > 0$)—then the manifold has a compatible C^∞ differentiable structure, that is, new charts may be found to make it C^∞. It is for that reason that I deemphasize the various degrees of differentiability and focus primarily on the smooth or C^∞ case.

(5) There is an undue amount of emphasis in Definition 1.2 on the choice of the open cover of the manifold. After we consider mappings between smooth manifolds, we will be able to introduce a natural notion of equiva-

lence between manifolds, and then we shall know when manifolds are equivalent, regardless of these covers.

Definition 1.3 Let M^n and N^k be smooth manifolds and $f: M^n \to N^k$ be a continuous map. Then f is called smooth (or on occasion just differentiable) if for every $x \in M^n$ and every pair of coordinate charts $\phi_\alpha: O_\alpha \to \mathbb{R}^n$, with $x \in O_\alpha$, and $\psi_\beta: U_\beta \to \mathbb{R}^k$, with $f(x) \in U_\beta$, on these manifolds, the composite mapping $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ has continuous partial derivatives of all orders at the point $\phi_\alpha(x)$.

Remarks (1) The mapping $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is easily seen to be defined on an open neighborhood of $\phi_\alpha(x)$, so it makes sense to talk about partial derivatives. In addition, one easily checks that this is independent of coordinate charts.

(2) In Definition 1.3, if we require that there be continuous partial derivatives of orders less than or equal to some nonnegative integer k, then the map f would be called C^k. As before, smooth is then the same as C^∞.

(3) The intuitive content of this definition is that we use the coordinate charts to transfer the notion from manifolds to the easily understood notion in Euclidean spaces.

Before looking into diffeomorphisms, submanifolds, etc., I would like to mention that there is one further type of manifold, which one might encounter in the literature. This is an analytic (or, rarely, C^ω) manifold. To define such manifolds, one requires that the maps in Definition 1.2

$$(\phi_\alpha | O_\alpha \cap O_\beta) \circ (\phi_\beta | O_\alpha \cap O_\beta)^{-1}$$

be given by convergent power series in n variables in a neighborhood of any point of their domains.

Similarly, we have an analytic map between analytic manifolds whenever the map $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is always represented by convergent power series (k series, each in n variables) in a neighborhood of any such point $\phi_\alpha(x)$.

It is a generally accepted notion that a map between two mathematical objects is an *isomorphism* or *equivalence* if there is a map going in the reverse direction that when composed with the original map—on either side—yields the identity map. This is also the idea behind a diffeomorphism.

Definition 1.4 Two smooth manifolds M_1^n and M_2^n are *diffeomorphic* (or occasionally equivalent) if there are smooth maps

$$f: M_1^n \to M_2^n \qquad \text{and} \qquad g: M_2^n \to M_1^n$$