Generalized Functions, Volume 5

Integral Geometry and Representation Theory

I. M. Gel′fand
M. I. Graev
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Integral Geometry and Representation Theory

Translator's Note

This English translation of the fifth volume of Professor Gel'fand's series on generalized functions contains all the material of the Russian fifth volume with the exception of its appendix. This appendix, in which generalized functions of a complex variable are discussed, appears as Appendix B of the first volume in the English translation.

The text of the translation does not deviate significantly from the Russian, although some minor typographical errors have been corrected and some equations have been renumbered. The symbol # has been used to indicate the end of the Remarks (set in small type in the Russian).

The subjects discussed in this book are often of interest both to mathematicians and to physicists, and each discipline has its own terminology. An attempt has been made to keep to the mathematicians' terminology, but some confusion is inevitable. I will appreciate suggestions for improvement in terminology and notation.

I wish to express my gratitude to the members of the Department of Mathematics at Northeastern University who have helped with the terminology. I am especially grateful to Professors Flavio Reis and Robert Bonic. I also wish to thank Dr. Eric H. Roffman and Professor E. C. G. Sudershan, who read the manuscript and galley proof and offered many helpful suggestions.

E. J. S.

Foreword

Originally the material in this book had been intended for some chapters of Volume 4, but it was later decided to devote a separate volume to the theory of representations. This separation was based on a suggestion by G. F. Rybkin, to whom the authors express their deep gratitude, for it is in excellent accord with the aims of the entire undertaking.

The theory of representations is a good example of the use of algebraic and geometric methods in functional analysis, in which transformations are performed not on the points of a space, but on the functions defined on it.

As we proceeded in our study of representation theory, we began to recognize that this theory is based on what we shall call integral geometry [see Gel'fand and Graev(9)]. Essentially, we shall understand integral geometry to involve the transition from functions defined on one set of geometrical objects (for instance on the points of some linear surface) to functions defined on some other set (for instance on the lines generating this surface).* Stated in this way, integral geometry is of the same general nature as classical geometry (Plücker, Klein, and others), in which new homogeneous spaces are formed out of elements taken from an originally given space. In integral geometry, however, we shall deal with such problems in what perhaps may be called their modern aspect: the transition from one space to the other shall be accomplished with the simultaneous transformation of the functions defined on it. This may be compared to the difference between classical and quantum mechanics: the transformations in classical mechanics are point transformations, while those of quantum mechanics are transformations in function space. (See the introduction to Chapter II.)

We have presumed to devote an entire volume to these elegant special problems in order to emphasize particularly this modern point of view relating geometry to functional analysis, as well as to point out the algebraic-geometric approach to functional analysis, an approach still in its earliest development.

In this book we shall not attempt a complete description of the theory of representations, for that would probably take several such volumes. Instead we shall restrict ourselves to the group of two-dimensional

* The term "integral geometry" as we use it here differs from its traditional meaning in which it involves calculating invariant measures on homogeneous spaces.

complex matrices of determinant one, which is of interest for many reasons. First, it is the simplest noncommutative and noncompact group. Further, it is the transformation group of many important spaces. In particular, it is locally isomorphic to the group of Lobachevskian motions, to the group of linear-fractional transformations of the complex plane, and several others. Finally, it is important in physics, for it is locally isomorphic to the proper Lorentz group.

The method we use in this book to develop the representation theory is not the only one possible. We have chosen the most natural approach, one based on the theory of generalized functions and making use of the excellent work of Bruhat(4). In this approach many of the phenomena of representation theory, in particular the relation between finite and infinite dimensional representations, become somewhat easier to understand.

This volume can be read almost independently of the previous ones. We assume only a knowledge of Chapters I and II and some of Chapter III of Volume 1, as well as their extension to the complex domain as discussed in Appendix B of that volume. The authors apologize beforehand for the incompleteness of the present volume. We hope that its underlying point of view will nevertheless be useful for those who are interested in new developments in functional analysis. The book is written to be read in one of two possible ways. Readers interested only in integral geometry may study Chapters I, II, and V, which are concerned only with integral geometry and are independent of the rest of the book. On the other hand readers interested only in representation theory can start with Chapter III, although an outline of the problems discussed is already given in Section 2 of Chapter II.

Chapters I and II were written by Gel'fand and Graev. The rest of the book was written by the three authors together. It contains a rewritten and expanded version of chapters originally written for Volume 4 by Gel'fand and Vilenkin (Chapters III and IV of this volume).

The authors express their deep gratitude to A. A. Kirillov and F. V. Shirokov, who read over the manuscript and made many helpful observations. They are especially grateful to L. I. Kopeykina whose help in the final stages of the manuscript greatly accelerated the publication of the book, and to S. A. Vilenkina for important help in the manuscript stage.

I. M. GEL'FAND
M. I. GRAEV
1962 N. YA. VILENKIN

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