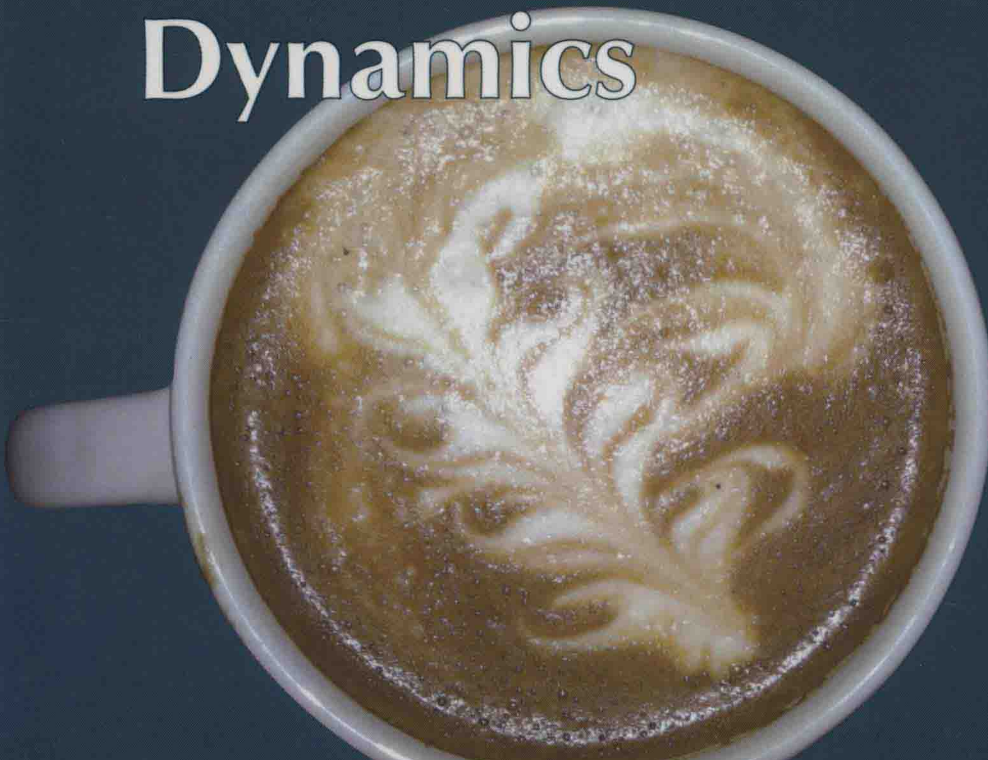
# Applied and Computational Measurable Dynamics



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Mathematical Modeling and Computation

# Applied and Computational Measurable Dynamics

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# Applied and Computational Measurable Dynamics

# Mathematical Modeling and Computation

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I would like to thank my boys,
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and especially my wife Elizabeth,
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**Naratip Santitissadeekorn**

# Preface

Measurable dynamics has traditionally referred to ergodic theory, which is in some sense a sister topic to dynamical systems and chaos theory. However, the topic has until recently been a highly theoretical mathematical topic which is generally less obvious to those practitioners in applied areas, who may not find obvious links to practical, real-world problems. During the past decade, facilitated by the advent of high-speed computers, it has become practical to represent the notion of a transfer operator discretely but to high resolution thanks to rapidly developing algorithms and new numerical methods designed for the purpose. An early book on this general topic is *Cell-to-Cell Mapping: A Method of Global Analysis for Nonlinear Systems* [167] from 1987.[1] A tremendous amount of progress and sophistication has come to the empirical perspective since then.

Rather than discussing the behaviors of complex dynamical systems in terms of following the fate of single trajectories, it is now possible to empirically discuss global questions in terms of evolution of density. Now complementary to the traditional geometric methods of dynamical systems transport study, particularly by stable and unstable manifold structure and bifurcation analysis, we can analyze transport activity and evolution by matrix representation of the Frobenius–Perron transfer operator. While the traditional methods allow for an analytic approach, when they work, the new and fast-developing computational tools discussed here allow for detailed analysis of real-world problems that are simply beyond the reach of traditional methods. Here we will draw connections between the new methods of transport analysis based on transfer operators and the more traditional methods. The goal of this book is not to become a presentation of the general topic of dynamical systems, as there are already several excellent textbooks that achieve this goal in a manner better than we can hope. We will bring together several areas, as we will draw connections between topological dynamics, symbolic dynamics, and information theory to show that they are also highly relevant to the Ulam–Galerkin representations. In these parts of the discussion, we will compare and contrast notions from topological dynamics to measurable dynamics, the latter being the first topic of this book. That is, if measurable dynamics means a discussion of a dynamical system in consideration of how much, how big, and other notions that require measure structure to discuss transport rates, topological dynamics can be considered as a parallel topic of study that asks similar questions in the absence of a measure that begets scale. As such, the mechanism and geometry of transport are more the focus. Therefore, including a discussion of topological dynamics in our primary discussion here on measurable dynamics should be considered complementary.

[1] Recent terminology has come to call these "set oriented" methods.

There are several excellent previous related texts on mathematical aspects of transfer operators which we wish to recommend as possible supplements. In particular, Lasota and Mackay [198] give a highly regarded discussion of the theoretical perspective of Frobenius–Perron operators in dynamical systems, whose material we overlap in as far as we need these elements for the computational discussion here. Boyarsky and Gora [50] also give a sharp presentation of an ensembles density perspective in dynamical systems, but more specialized for one-dimensional maps, and some of the material and proofs therein are difficult to find elsewhere. Of course the book by Baladi [11] is important in that it gives a thoroughly rigorous presentation of transfer operators, including a unique perspective. We recommend highly the book by Zhou and Ding, [324], which covers a great deal of theoretical information complementary to the work discussed in this book, including Ulam's method and piecewise constant approximations of invariant density, piecewise linear Markov models, and especially analysis of convergence. Also an in-depth study can be found concerning connections of the theory of Frobenius–Perron operators and the adjoint Koopman operator, as well as useful background in measure theory and functional analysis. The book by McCauley [215] includes a useful perspective regarding what is becoming a modern perspective on computational insight into behaviors of dynamical systems, especially experimentally observed dynamical systems. That is, finite realizations of chaotic data can give a great deal of insight. This is a major theme which we also develop here toward the perspective that a finite time sample of a dynamical system is not just an estimate of the long time behavior, as suggested perhaps by the traditional perspective, but in fact finite time samples are most useful in their own right toward understanding finite time behavior of a dynamical system. After all, any practical, real-world observation of a dynamical system can be argued to exist only during a time window which cannot possibly be infinite in duration.

There are many excellent textbooks on the general theory of dynamical systems, clearly including Robinson [268], Guckenheimer and Holmes [146], Devaney [95], Alligood, Sauer, and Yorke [2], Strogatz [301], Perko [251], Meiss [218], Ott [244], Arnold [4], Wiggins [316], and Melo and van Strein [89], to name a few. Each of these has been very popular and successful, and each is particularly strong in special aspects of dynamical systems as well as broad presentation. We cannot and should not hope to repeat these works in this presentation, but we do give what we hope is enough background of the general dynamical systems theory in order that this work can be somewhat self-contained for the nonspecialist. Therefore, there is some overlap with other texts insofar as background information on the general theory is given, and we encourage the reader to investigate further in some of the other cited texts for more depth and other perspectives. More to the point of the central theme of this textbook, the review article by Dellnitz and Junge [87] and then later the Ph.D. thesis by Padberg [247] (advised by Dellnitz) both give excellent presentations of a more computationally based perspective of measurable dynamical systems in common with the present text, and we highly recommend them. A summary of the German school's approach to the empirical study of dynamical systems can be found in [112], and [82]. Also, we recommend the review by Froyland [121]. Finally, we highly recommend the book by Hsu [167], and see also [166], which is an early and less often cited work in the current literature, as we rarely see "cell-to-cell mappings" cited lately. While lacking the transfer oriented formalism behind the analysis, this cell-to-cell mapping paradigm is clearly a precursor to the computational methods which are now commonly called set oriented methods. Also, we include a discussion and contrast to the early ideas by Ulam [307] called the Ulam method. Here we hope to give a useful broad presentation in a manner that includes some necessary background to allow a sophisticated but otherwise not specialized student or researcher to dive into this topic.

**Acknowledgments.** Erik Bollt would like to thank the several students and colleagues for discussions and cooperation that have greatly influenced the evolution of his perspective on these topics over several years, and who have made this work so much more enjoyable as a shared activity. He would also like to thank the National Science Foundation and the Office of Naval Research and the Army Research Office, who have supported several aspects of this work over the recent decade.

# Contents

# Chapter 1

# Dynamical Systems, Ensembles, and Transfer Operators

## 1.1 Ergodic Preamble

In this chapter, we present the heuristic arguments leading to the Frobenius–Perron operator, which we will restate with more mathematical rigor in the next chapter. This chapter is meant to serve as a motivating preamble, leading to the technical language in the next chapter. As such, this material is a quick start guide so that the more detailed discussion can be followed with more motivation. It also provides enough background so that the techniques in subsequent chapters can be understood without necessarily reading all of the mathematical theory in the middle chapters.

In terms of practical application, the field of measurable dynamics has been hidden in a forest of formal language of pure mathematics that may seem impenetrable to the applied scientist. This language may be quite necessary for mathematical proof of the methods in the field of ergodic theory. However, the proofs often require restricting the range of problems quite dramatically, whereas the utility may extend quite further. In reality, the basic tools one needs to begin practice of measurable dynamics by transfer operator methods are surprisingly simple, while still allowing useful studies of transport mechanisms in a wide array of real-world dynamical systems. It is our primary goal to bring out the simplicity of the field for practitioners. We will attempt to highlight the language necessary to speak properly in terms necessary to prove convergence, invariance, steady state, and several of the other issues rooted in the language of ergodic theory. But above all, we wish to leave a spine of simple techniques available to practitioners from outside the field of mathematics. We hope this book will be useful to those experimentalists with real questions coming from real data, and to any students interested in such issues.

Our discussion here may be described as a contrast between the Lagrangian perspective of following orbits of single initial conditions and the Eulerian perspective associated with the corresponding dynamical system of the transfer operator which describes the evolution of measurable ensembles of initial conditions while focusing at a location. This leads to issues traditionally affiliated with ergodic theory, a field which has important practical implications in the applied problems of transport study that are of interest here. Thus we hope the reader will agree that both perspectives allow important information to be derived from a dynamical system. In particular, the transfer operator approach will allow us to discuss

- exploring global dynamics and characterization of the global attractors,
- estimating invariant manifolds,
- partitioning the phase space into invariant regions, almost invariant regions, and coherent sets,
- rates of transport between these partitioned regions,
- decay of correlation,
- associated information theoretic descriptions.

As we will discuss throughout this book, the question of transport can be boiled down to a question of walks in graphs, stochastic matrices, Markov chains, graph partitioning questions, and matrix analysis, together with Galerkin's methods for discussing the approximation. We leave this section with a picture in Fig. 1.1, which in some sense highlights so many of the techniques in the book. We will refer back to this figure often throughout this text. For now, we just note that the figure is an approximation of the action on the phase space of a Henon mapping as the action of a directed graph. The Henon mapping,

$$\begin{aligned} x_{n+1} &= y_{n+1} - a x_n^2, \\ y_{n+1} &= b x_n, \end{aligned} \tag{1.1}$$

for parameter values $a = 1.4$, $b = 0.3$, is frequently used as a research tool and as a pedagogical example of a smooth chaotic mapping in the plane. It is a diffeomorphism that highlights many issues of chaos and chaotic attractors in more than one dimension. Such mappings are not only interesting in their own right, but they also offer a step toward understanding differential equations by Poincaré section mappings.

## 1.2 The Ensemble Perspective

The dynamical systems point of view is generally Lagrangian, meaning that we focus on following the fate of trajectories corresponding to the evolution of a single initial condition. Such is the perspective of an ODE, Eq. (2.1), as well as a map, Eq. (2.7). Here we contrast the Lagrangian perspective of following single initial conditions to the Eulerian perspective rooted in following measurable ensembles of initial conditions, based on the associated dynamical system of transfer operators and leading to ergodic theory. We are most interested here in the transfer operator approach in that it may shed light on certain applied problems to which we have already alluded and we will detail.

**Example 1.1 (following initial conditions, the logistic map).** The logistic map,

$$x_{n+1} = L(x_n) = 4x_n(1 - x_n), \tag{1.2}$$

is a model of resource limited growth in a population system. The logistic map is an extremely popular model of chaos, partly for pedagogical reasons of simplicity of analysis,