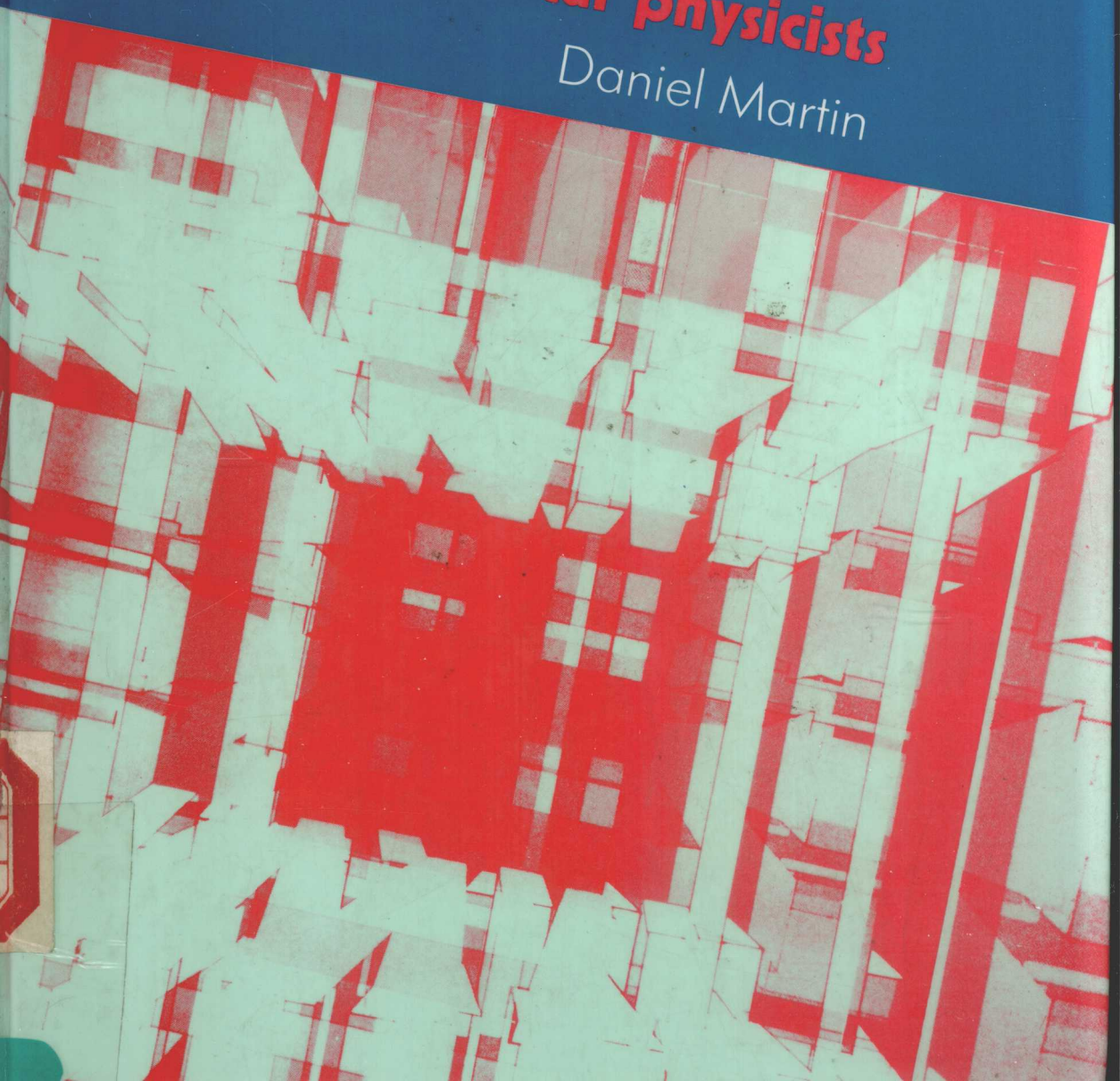


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*an introduction for
mathematical physicists*

Daniel Martin



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MANIFOLD THEORY

An Introduction for Mathematical Physicists

DANIEL MARTIN M.A., B.Sc., Ph.D.

Department of Mathematics, University of Glasgow



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To the memory of my mother Rose Martin

Preface

The aim of this book, which is based on courses of lectures given over many years to first-year research students in theoretical physics in this university, is to give a comprehensive account of basic manifold theory suitable for students who have completed an honours degree in physics or in mathematics and are about to begin postgraduate work in theoretical physics, in particular, relativity. The treatment is rigorous but less condensed (and therefore I hope easier to follow) than that given in some of the books written primarily for pure mathematicians and demanding considerable mathematical maturity on the part of the reader. For various reasons the proofs of a few theorems, especially among those in the latter part of the book have been omitted. Since the prerequisites for the study of manifold theory include a knowledge of certain parts of linear algebra not normally included in a first course on that subject either for mathematicians or for physicists, the book begins with a chapter dealing with these matters. Likewise, two appendices on topology have been included for the benefit of those readers who have not been exposed to an undergraduate course on the subject. Throughout the book numerous examples both worked and unworked have been provided. Any student who has conscientiously worked through the book should be well prepared for a study of the larger and more specialist books on manifold theory and related subjects.

In writing my lecture notes I was, of course, greatly influenced by the standard books on the subject and I acknowledge here my indebtedness to them, and, indeed, to all but a few of the books listed in the Bibliography. Where I have leaned heavily on a particular book, I hope that I have made that clear in the text. Finally, I wish to thank my colleagues Dr E. A. McHarg and Dr J. B. Hickey for valuable comments on certain chapters of the book, and Mr M. Shardlow of Ellis Horwood Ltd for the care that he has taken with the manuscript.

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Table of contents

Preface	9
1 Vector spaces	
1.1 Change of basis	11
1.2 Inner product spaces	15
1.3 Affine spaces	19
1.4 Minkowski space-time	20
Exercises 1	25
2 Tensor algebra	
2.1 Tensors	26
2.2 p -forms	41
2.3 Tensors in an inner product space	51
Exercises 2	58
3 Differentiable manifolds	
3.1 Notes on advanced calculus	60
3.2 Differentiable manifolds	62
3.3 The topology of a differentiable manifold	67
3.4 Orientable manifolds	72
3.5 Product manifolds	76
3.6 Quotient manifolds	77
3.7 Differentiable mappings	83
Exercises 3	86
4 Vector and tensor fields on a manifold	
4.1 Tangent vectors and spaces to a manifold	88
4.2 The differential of a mapping	93
4.3 Vector fields on a manifold	104
4.4 Tensor fields on a manifold	109
Exercises 4	113

5 Exterior differential forms

5.1 Exterior forms on a manifold 114

5.2 Exterior differentiation 118

5.3 The interior product 128

5.4 de Rham cohomology 131

5.5 Application to vector calculus 136

5.6 Maxwell’s equations of the electromagnetic field 139

Exercises 5 142

6 Differentiation on a manifold

6.1 Introduction 145

6.2 The Lie derivative 145

6.3 Affine connexions and covariant differentiation 154

6.4 Parallel transport 166

6.5 The curvature and torsion fields associated with an affine connexion . 167

6.6 Curvature and torsion forms 174

Exercises 6 177

7 Pseudo-Riemannian and Riemannian manifolds

7.1 The metric tensor 180

7.2 Products and warped products of manifolds 184

7.3 Differentiation on a pseudo-Riemannian manifold 186

7.4 Geodesics 203

7.5 Riemannian curvature 217

7.6 Computation of the curvature tensor 223

7.7 Killing vector fields 228

7.8 The dual of a tensor 233

Exercises 7 235

8 Symplectic manifolds

8.1 Symplectic manifolds 239

8.2 Hamiltonian vector fields 240

8.3 The symplectic group 244

Exercises 8 244

9 Lie groups

9.1 Lie groups 245

9.2 The Lie algebra of a Lie group 252

9.3 The action of a Lie group on a manifold 259

9.4 The exponential of a matrix 266

9.5 The Lie algebras of some subgroups of $GL(n, \mathbb{R})$ and of $GL(n, \mathbb{C})$. 272

9.6 The adjoint algebra of a Lie algebra 278

Exercises 9 282

10 Integration on a manifold

10.1	Introduction	285
10.2	Integration of an n -form over a manifold of dimension n	286
10.3	Integration of a function over a pseudo-Riemannian manifold	289
10.4	Manifolds with boundary and Stokes' theorem	290
	Exercises 10	301

11 Fibre bundles

11.1	Fibre bundles	303
11.2	Principal fibre bundles.	315
11.3	Constructions involving fibre bundles	317
11.4	Connexions on a principal fibre bundle	319
11.5	Curvature	329
11.6	Parallelism and the holonomy group.	340
11.7	Linear connexions	344
11.8	Note on Chern classes.	347
	Exercises 11	348

12 Complex linear algebra. Almost complex manifolds

12.1	Complex linear algebra	349
12.2	Almost complex manifolds.	355
12.3	Hermitian manifolds.	358
12.4	Kählerian manifolds	361
12.5	Covariant differentiation on a Hermitian manifold	364
	Exercises 12	373

Appendix 1: Analytic topology

A1.1	Naive set topology.	375
A1.2	Metric spaces	377
A1.3	General topological spaces	383
A1.4	Special topological spaces	389
A1.5	Topological groups	394

Appendix 2: Quaternions and Cayley numbers

A2.1	Quaternions	396
A2.2	Cayley numbers (or octonions).	398

Appendix 3: The semidirect product of two groups 400**Appendix 4: Homotopy review 404****Bibliography 409****Some answers, some hints and some fragmentary solutions to the exercises . . . 412****Index 419**

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