# THE NATURE OF MATHEMATICS

KARL J. SMITH

7TH EDITION

# The Nature of Mathematics

## Seventh Edition

**Karl J. Smith**

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*To love is to admire with the heart;*
*to admire is to love with the mind.*

***I dedicate this book with love***
***to my daughter, Melissa,***
***and the newest member of my family, Benjamin,***
***on the occasion of their marriage.***

*December* 1994

# Preface

This edition marks the 20th anniversary of the publication of this book. As I reflect on the changes of content over those years, I am impressed with the direction of liberal arts mathematics. Textbooks necessarily reflect current trends in education, and in the case of liberal arts mathematics courses, I have seen a movement from a complete survey course to a course that attempts to equip students with the necessary skills of mathematics literacy to function effectively in our technological world. As the demands of society change, so do the essential competencies needed by individuals to live productively in that society. *All* students will need competence in essential areas of mathematics.

In recognizing this need, most colleges and universities have some type of a mathematics requirement or a requirement that students must demonstrate mathematics competency *regardless of their academic major.* Those students in mathematics or the sciences are often required to take a great deal of mathematics, whereas those students in the arts or soft sciences will generally need one mathematics course, and perhaps a statistics course. This book was written for those students who need a mathematics course to satisfy the general university competency requirement in mathematics.

Because of the university requirement, many students enrolling in a course that uses my book have postponed taking this course as long as possible, are dreading the experience, and are coming with a great deal of anxiety. I wrote this book with one overriding goal: to create a positive attitude toward mathematics. Rather than simply presenting the technical details needed to proceed to the next course, I have attempted to give insight into what mathematics is, what it accomplishes, and how it is pursued as a human enterprise. However, at the same time, I have included in this seventh edition a great deal of material to help students estimate, calculate, and solve problems *outside* the classroom or textbook setting.

I frequently encounter people who tell me about their unpleasant experiences with mathematics. I have a true sympathy for those people, and I recall one of my elementary school teachers who assigned additional arithmetic problems as punishment. This can only create negative attitudes toward mathematics, which is indeed unfortunate. If elementary school teachers and parents have positive attitudes toward mathematics, their children cannot help but see some of the beauty of the subject. I want students to come away from this course with the feeling that mathematics can be pleasant, useful, and practical—and enjoyed for its own sake.

The prerequisites for this course vary considerably, as do the backgrounds of students who enroll in the course. Some schools have no prerequisites, but other schools have an intermediate algebra prerequisite. The students, as well, have heterogeneous backgrounds. Some have little or no mathematics skills; others have had a great deal of mathematics. Even though the usual prerequisite for using this book is intermediate algebra, a careful selection of topics and chapters would allow a class with a beginning algebra prerequisite to study effectively from this book.

This book was written to meet the needs of all of these students and schools. How did I accomplish that goal? First, the chapters are almost independent of one another, and can be covered in any order appropriate to a particular audience. Second, the problems are designed to be the core of the course. There are problems that every student will find easy and provide the opportunity for success; there are also problems that are very challenging. Much interesting material appears in the problems, and students should get into the habit of reading (not necessarily working) all the problems whether they are assigned or not.

**A Problems:** mechanical or drill problems

**B Problems:** require understanding of the concepts

**Problem Solving Problems:** require problem-solving skills or original thinking

**Individual Research:** requires research or library work

**Group Research:** requires not only research or library work, but also group participation (See the index for a list of group projects.)

The major themes of this book are problem solving and estimation in the context of presenting the great ideas in the history of mathematics. I believe that *learning to solve problems is the principal reason for studying mathematics.* Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations. Solving word problems in most textbooks is one form of problem solving, but students also should be faced with non–text-type problems. In the first section of this edition I introduce students to Polya's problem-solving techniques, and these techniques are used throughout the book to solve non–text-type problems. These problem-solving examples are found throughout the book (marked as **Polya's Method** examples). Also new to this edition are problems in *each* section that require Polya's method for problem solving.

*Students should learn the language and notation of mathematics.* Most students who have trouble with mathematics do not realize that mathematics *does require hard work.* The usual pattern for most mathematics students is to open the book to the assigned page of problems, and begin working. Only after getting "stuck" is an attempt made to "find it in the book." The final resort is reading the text. In this book the student is asked not only to "do math problems," but also to "experience mathematics." This means it is necessary to become involved with the **concepts** being presented, not "just get answers." In fact, the slogan "Mathematics Is Not a Spectator Sport" is not just an advertising slogan, but an invitation which suggests that the only way to succeed in mathematics is to become involved with it. Students will learn to receive mathematical ideas through listening, reading, and visualizing. They are expected to present mathematical ideas by speaking, writing, drawing pictures and graphs, and demonstrating with concrete models. New to this edition is a category of problems in each section which is designated **In Your Own Words,** which provides practice in communication skills.

## A Personal Note

Writing a mathematics textbook is both enjoyable and challenging. To make mathematics come alive, I have included many items not usually found in a textbook. For example, I've included cartoons and quotations, and have used the margins for news clippings and historical notes. The historical notes are not strictly biographical reports, but instead focus on the people to convey some of the humanness of mathematics. Nearly every major mathematician (and many minor ones) has some part of his or her life to tell on the pages of this book. At the end of each chapter there is an interview of some living mathematician. I sat down and made a list of those persons who are the most famous, or those whom I greatly respect. I did not know how my request for an interview would be received, but to my surprise each of these persons was most gracious in providing me not only with biographical information, but with personal details of their lives so that I could share some of their humanness with users of this book. I treasure the correspondence I had with these people.

## A Note for Instructors

Feel free to arrange the material in a different order from that presented in the text. I have written the chapters to be as independent of one another as possible. There is much more material in this book than could be covered in a single course. The usual audience is liberal arts, teacher training, finite mathematics, college algebra, or, at many smaller schools, a combination of these.

If you would like to have a copy of my classroom handout on term projects, please write to me at Brooks/Cole Publishing Company and I will send it to you.

I have written an extensive *Instructor's Manual* to accompany this book. It includes the complete solutions to all the problems (including the "Problem Solving" problems) as well as teaching suggestions and transparency masters. For those who wish to integrate the computer into the entire course, there are computer problems in both BASIC and LOGO to accompany each chapter.

George Bradley has written a manual, *Problem Solving with Creative Mathematics,* to complement the material of this text. Topics are introduced using group and individual projects that can be done in or outside of class. Students discover many of the basic principles themselves, learn to work cooperatively, and then are required to effectively communicate what they have learned.

Also available are sample tests, not only in hard copy form, but also in electronic form for both IBM and Macintosh formats. For those students who are preparing to take Florida's College Level Academic Test (CLAST), there is a study manual that is indexed to topics in this text. This manual includes explanations of each of the 56 skills specified by the State of Florida for inclusion on the computational portion of the CLAST. It also includes more than 1,000 CLAST-style multiple-choice exercises and worked examples (including three complete sample CLAST tests). It includes test-taking advice geared specifically to the CLAST, but also includes valuable guidelines for all students.

## Changes from the Previous Edition

It is necessary to be very careful about the changes to a successful book. I did not want to change the book so much that I would lose users of the current edition. On the other hand, I felt it was time for a revision, and I wanted to offer many new ideas.

Each chapter of this edition begins with a feature I call **In the Real World.** The chapter begins with a dialogue and usually poses some problem or question. By the end of the chapter we will have learned the necessary skills to discuss the solution to the problem, which is presented at the end of each chapter as a **Commentary.** This feature then concludes with a **Challenge** that I think students will find interesting.

The seventh edition has three new chapters: Chapter 3, The Nature of Calculation, discusses numeration systems and the history of calculating devices. It also takes the opportunity to discuss calculators. It is recommended that those using this book have an inexpensive calculator. Chapter 8, The Nature of Measurement, considers precision, accuracy, estimation, area, volume, and capacity. Chapter 12, The Nature of Mathematical Systems, includes solving linear equations and inequalities. In addition, Chapter 1 has been completely rewritten to include a much clearer presentation of problem solving. The content of Chapter 3 has been distributed to several places throughout the book, reflecting my belief that computers are *tools* that should be utilized to help us solve problems. New topics in this edition include cryptography; evaluation, variables, and spreadsheets; topology and fractals; precision, accuracy, and estimation; sampling; a section discussing the nature of calculus; as well as a section on limits. Problem-solving sections have been added to Chapters 1, 2, 5, 6, and 12.

The book has been redesigned to give a less cluttered appearance. The use of color greatly enhances the presentation of more material with less clutter. Motivational material has been consolidated into Historical Notes and News Clips. Author notes have been incorporated into the textual development. Examples, new ideas, and tables are now clearly identified.

I have also spent a great deal of effort in pulling together the information on the endpapers. I do not intend that these be decorative, but rather very informative in putting the great ideas of mathematics into a historical perspective.

## Acknowledgments

I would like to thank Diana Gerardi for her valuable suggestions in improving my book. I also appreciate the suggestions of the reviewers of this edition: Jeffrey Allbritten, Middle Tennessee State University; Charles Baker, West Liberty State College; Mark Christie, Louisiana State University; Wil Clarke, La Sierra University; Penelope Ann Coe, Central Connecticut State University; Gentil Estevez, Interamerican University; Robert Fliess, West Liberty State College; Gary Gislason, University of Alaska–Fairbanks; Mark Greenhalgh, Fullerton College; Michael Helinger, Clinton Community College; Michael Jones, New York Institute of Technology and Suffolk Community College; Carol McVey, Florence–Darlington Technical College; Michael Pinter, Belmont University; Richard Rempel, Bethel College; Jane Rood, Eastern Illinois University; Andrew Simoson, King College; and James Walters, Cape Fear Community College.

One of the nicest things about writing a successful book is all of the letters and suggestions I've received. I would like to thank the following people who gave suggestions for previous editions of this book: Peter R. Atwood, John August, Jerald T. Ball, Carol Bauer, George Berzsenyi, Jan Boal, Kolman Brand, Chris C. Braunschweiger, T. A. Bronikowski, Charles M. Bundrick, T. W. Buquoi, Eugene Callahan, Michael W. Carroll, Joseph M. Cavanaugh, James R. Choike, Gerald Church, Lynn Cleaveland, Thomas C. Craven, Gladys C. Cummings, Ralph De Marr, Maureen Dion, Charles Downey, Mickle Duggan, Samuel L. Dunn, Beva Eastman, William J. Eccles, Ernest Fandreyer, Loyal Farmer, Gregory N. Fiore, Richard Freitag, Gerald E. Gannon, Ralph Gellar, Mark Greenhalgh, Martin Haines, Abdul Rahim Halabieh, John J. Hanevy, Robert L. Hoburg, Caroline Hollingsworth, Scott Holm, Libby W. Holt, Peter Hovanec, M. Kay Hudspeth, Carol M. Hurwitz, James J. Jackson, Vernon H. Jantz, Charles E. Johnson, Nancy J. Johnson, Martha C. Jordan, Judy D. Kennedy, Linda H. Kodama, Daniel Koral, Helen Kriegsman, C. Deborah Laughton, William Leahey, John LeDuc, William A. Leonard, Adolf Mader, John Martin, Cherry F. May, George McNulty, Charles C. Miles, Allen D. Miller, John Mullen, Charles W. Nelson, John Palumbo, Gary Peterson, Michael Petricig, James V. Rauff, Paul M. Riggs, Peter Ross, O. Sassian, Mickey G. Settle, James R. Smart, Glen T. Smith, Donald G. Spencer, Gustavo Valadez-Ortiz, John Vangor, Arnold Villone, Clifford H. Wagner, Barbara Williams, Stephen S. Willoughby, and Bruce Yoshiwara.

I would especially like to thank my editor, Robert J. Wisner of New Mexico State, for his countless suggestions and ideas; Craig Barth, Jeremy Hayhurst, Paula Heighton, Gary W. Ostedt, and Joan Marsh of Brooks/Cole; as well as Jack Thornton, for the sterling leadership and inspiration he has been to me from the inception of this book to the present.

The production of this book was a true team effort, and I especially appreciate Susan Reiland for her help in countless ways, including editing, accuracy checking, and giving me tireless support and help. I would also like to thank Kathi Townes, Brian Betsill, and Stephanie Kuhns at TECHarts for their long hours and superb work.

Finally, my thanks go to my wife, Linda, whose suggestions and help were invaluable. Without her this book would exist only in my dreams, and I would have never embarked as an author.

Karl J. Smith
Sebastopol, CA

# To the Student

## A Fable

Once upon a time, two young ladies, Melinda and Sheila, came to a town called Mathematics. People had warned them that this is a particularly confusing town. Many people who arrived in Mathematics were very enthusiastic, but could not find their way around, became frustrated, gave up, and left town.

Melinda was strongly determined to succeed. She was going to learn her way through the town. For example, to learn how to go from her dorm to class, she concentrated on memorizing this clearly essential information: she had to walk 325 steps south, then 253 steps west, then 129 steps in a diagonal (southwest), and finally 86 steps north. It was not easy to remember all of that, but fortunately she had a very good instructor who helped her to walk this same path 50 times. To stick to the strictly necessary information, she ignored much of the beauty along the route, such as the color of the adjacent buildings or the existence of trees, bushes, and nearby flowers. She always walked blindfolded. After repeated exercising, she succeeded in learning her way to class and also to the cafeteria. But she could not learn the way to the grocery store, the bus station, or a nice restaurant; there were just too many routes to memorize. It was so overwhelming! Finally she gave up and left town; Mathematics was too complicated for her.

Sheila, on the other hand, was of a much less serious nature. She did not even intend to memorize the number of steps of her walks. Neither did she use the standard blindfold that students need for learning. She was always curious, looking at the different buildings, trees, bushes, and nearby flowers or anything else not necessarily related to her walk. Sometimes she walked down dead-end alleys to find out where they were leading, even if this was obviously superfluous. Curiously, Sheila succeeded in learning how to walk from one place to another. She even found it easy and enjoyed the scenery. She eventually built a building on a vacant lot in the city of Mathematics.*

* My thanks to Emilio Roxin of the University of Rhode Island for the idea for this fable.

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# THE NATURE OF MATHEMATICS

Seventh Edition

# The Nature of Problem Solving

# I

*The idea that aptitude for mathematics is rarer than aptitude for other subjects is merely an illusion which is caused by belated or neglected beginners.*

—J. F. Herbart

## *In the Real World . . .*

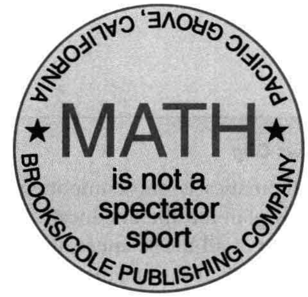
"Hey, Tom, what are you taking this semester?" asked Susan. "I'm taking English, history, and math. I can't believe my math teacher," responded Tom. "The first day we were there, she walked in, wrote her name on the board, and then she asked, 'What is the millionth counting number that is not the square or cube of a counting number?' Who cares!"

"Oh, I had that math class last semester," said Susan. "It isn't so bad. The whole idea is to give you the ability to solve problems *outside* the class. I want to get a good job when I graduate, and I've read that because of the economy, employers are looking for people with problem-solving skills. I hear that working smarter is more important than working harder."

# 1.1 PROBLEM SOLVING

## A Word of Encouragement

Do you think of mathematics as a difficult, foreboding subject that was invented hundreds of years ago? Do you think that you will never be able (or even want) to use mathematics? If you answered "yes" to either of these questions, then I want you to know that I have written this book for you. I have tried to give you some insight into how mathematics is developed, and to introduce you to some of the people behind the mathematics. In this book, I will present some of the great ideas of mathematics, and then we will look at how these ideas can be used in an everyday setting to build your problem-solving abilities. *The most important prerequisite for this course is an openness to try out new ideas—a willingness to experience the suggested activities rather than to sit on the sideline as a spectator.* I have attempted to make this material interesting by putting it together differently from the way you might have had mathematics presented in the past. You will find this book difficult if you wait for the book or the teacher to give you answers—instead *be willing to guess, experiment, estimate, and manipulate,* and try out problems *without fear of being wrong!* It will take a daily commitment on your part, and you will find mathematics difficult if you try to "get it done" in spurts.




There is a common belief that mathematics is to be pursued only in a clear-cut logical fashion. This belief is perpetuated by the way mathematics is presented in most textbooks. Often it is reduced to a series of definitions, methods to solve various types of problems, and theorems. These theorems are justified by means of proofs and deductive reasoning. I do not mean to minimize the importance of proof in mathematics, for it is the very thing that gives mathematics its strength. But the power of the imagination is every bit as important as the power of deductive reasoning. As the mathematician Augustus De Morgan once said, "The power of mathematical invention is not reasoning but imagination."

Mathematics is alive and constantly changing. As we complete the last decade of this century, we stand on the threshold of major changes in the mathematics curriculum in the United States.

**A Note for the Student**

There are many reasons for reading a book, but the best reason is because you want to read it. Although you are probably reading this first section because you were required to do so by your instructor, it is my hope that in a short while you will be reading this book because you *want* to read it. It was written for people who think they don't like mathematics, or people who think they can't work math problems, or people who think they are never going to use math. As you begin your trip through this book, I wish you a BON VOYAGE!

K J Smith

## Problem Solving

We begin this study of problem solving by looking at the *process* of problem solving. As a mathematics teacher, I often hear the comment, "I can do mathematics, but I can't solve word problems." There *is* a great fear and avoidance of "real-life" problems because they do not fit into the same mold as the "examples in the book." Few practical problems from everyday life come in the same form as those you study in school.

To compound the problem, learning to solve problems takes time. All too often, the mathematics curriculum is so packed with content that the real process of problem solving is slighted, and because of time limitations, becomes an exercise in mimicking the instructor's steps instead of developing into an approach that can be used long after the final examination is over.

Before we build problem-solving skills, it is necessary to build certain prerequisite skills necessary for problem solving. It is my goal to develop your skills in the mechanics of mathematics, in understanding the important concepts, and finally in applying those skills to solve a new type of problem. I have segregated the problems in this book to help you build these different skills:

| | |
|---|---|
| In Your Own Words | This type of problem asks you to discuss or rephrase main ideas or procedures using your own words. |
| A Problems | These are mechanical and drill problems. |
| B Problems | These problems require an understanding of the concepts. |
| Problem Solving | These require problem-solving skills or original thinking. |
| Problems for Research | These problems require research or library work. Most are intended for individual research but a few are group research projects. |

**News Clip**

Mathematics is one component of any plan for liberal education. Mother of all the sciences, it is a builder of the imagination, a weaver of patterns of sheer thought, an intuitive dreamer, a poet. The study of mathematics cannot be replaced by any other activity . . .

*American Mathematical Monthly*
Volume 56, 1949, p. 19

The model for problem solving that we will use was first published in 1945 by the great, charismatic mathematician George Polya. His book *How to Solve It* (Princeton University Press, 1973) has become a classic. In Polya's book you will find this problem-solving model as well as a treasure trove of strategy, know-how rules of thumb, good advice, anecdotes, history, and problems at all levels of mathematics. His problem-solving model is as follows.

**Guidelines for Problem Solving**

| | |
|---|---|
| First | You have to *understand the problem.* |
| Second | *Devise a plan.* Find the connection between the data and the unknown. Look for patterns, relate to a previously solved problem or a known formula, or simplify the given information to give you an easier problem. |
| Third | *Carry out the plan.* |
| Fourth | *Look back.* Examine the solution obtained. |