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ABRAHAM A. FRAENKEL

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Preface

THIS VOLUME is essentially a translation of the first part of my Hebrew book *Mavo LeMathematika*[1] (*Introduction to Mathematics*), but quite a number of modifications and additions have been incorporated.

Two volumes of similar size, nature, and purpose are planned for publication during the next few years. One will deal with the fundamental concepts of algebra (group, ring, field) and their role in the extension of the number concept to real, complex, and hypercomplex numbers; the other will present and discuss the theory of sets, in particular transfinite cardinal and ordinal numbers.

These volumes developed from talks in the adult education program given by the author in towns and rural settlements of Palestine (now Israel) from 1929 on. Consequently, the subject of the present volume and its treatment meet the needs, abilities, and interests of gifted high school students, of college freshmen, and, indeed, of laymen who are interested in knowing what mathematics really deals with—a question whose answer may have been concealed rather than revealed by the presentation in their classes.

I wish to express my sincere thanks to Professor Jekuthiel Ginsburg without whose efforts the publication of the volume would not have been possible.

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[1] Vol. I, Jerusalem, 1942; vol. II, Jerusalem, 1954. These two volumes contain five parts which deal with the following topics: integers and theory of numbers; the extension of the number-concept, including groups, rings, fields, and a survey of algebra; analysis; theory of sets; geometry. A supplementary volume which deals with the foundations of mathematics is in preparation.

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CHAPTER I

Natural Numbers as Cardinals

WE ARE here concerned with the nature of the positive integers which are known also as *natural numbers.* Contemporary mathematical opinion follows that of previous centuries in regarding these numbers as the keystone of the mathematical structure. In the words of Weyl: "Mathematics is entirely dependent, even with respect to the logical forms of its exposition, upon the nature of the natural numbers." However, unlike our predecessors of only half a century ago, most modern mathematicians do not think of mathematics as *beginning* with the natural numbers and proceeding thence to the development of various branches. The latest views rather tend to assign to integers a *middle* position in the structure of the science. The lower portions are devoted to the foundations of mathematics which are established in the general theories of relation, order, sets, groups, fields etc. as well as of logic, while the various mathematical disciplines, such as theory of numbers, algebra, theory of functions and also most parts of geometry, start from the level of natural number. It should be noted, however, that even today a significant group of mathematicians believe that it is impossible to develop the natural numbers from more fundamental concepts and that we must regard them as emerging from the very nature of the human mind or even as objects which are imposed upon us regardless of our will. This last view has been expressed in a famous dictum of Kronecker[1] (1823–91): God created the integers, the rest is work of man.

1. The Positional Notation

Before proceeding with a study of natural numbers it is appropriate to devote a few words to their *notation*, that is, to the numeral system. The difficulty involved in the notation of numbers arises from the fact that infinitely many numbers must be represented by means of a finite array of symbols. Moreover, we make the following two demands

[1] *See Math. Annalen,* Vol. 43, 1893, p. 15.

of our symbolism: (*a*) that the number of symbols be sufficiently small to avoid undue claims to the memory; (*b*) that the representation of even reasonably large numbers by means of our symbols should not occupy an amount of space which would make its notation inconvenient. Both these objectives were attained by Hindu mathematicians[2] through their invention of the positional system in which the value of a numeral varies with the different places in which it appears. Thus, if we employ a given number j as the base of the system, the numeral s represents the number s or $s \cdot j^0$ only when it occupies the initial place, that is, when it appears as the first digit to the right in the representation of the entire number. If the numeral occupies the second place (second digit from the right), it represents the number $s \cdot j (s \cdot j^1)$; etc. In the nth place it will represent $s \cdot j^{n-1}$. For example, $s_2 s_1 s_0$ denotes the number $s_0 + s_1 \cdot j + s_2 \cdot j^2$ on condition that s_0, s_1, s_2 are digits, i. e., numbers less than j. Hence the value of a symbol varies with its position only when the base j is greater than 1, since the powers of 1 are all equal. Therefore only such bases are admissible.

However, the positional principle in itself does not suffice. Were we to employ, in a system with the base j, only the numerals 1, 2, 3, . . ., $j - 1$, we would soon be confronted by numbers which could not be described by means of our system. If, for example, we consider our decimal system: $j = 10$, we should have to represent both thirteen and one hundred and three by the same symbol, 13. As a matter of fact, one of the most significant contributions to scientific progress was the *invention of the zero*, i. e., the principle that those places in which we do not wish to put a numeral must be occupied by a special numerical symbol, 0. This invention, without which the use of the positional notation would have been impossible, was brought to the west from India by Arab scholars. The genius involved in this invention can best be gauged from the fact that the greatest Greek mathematicians, including Archimedes (287–212 B.C.) and Apollonios (265?–170 B.C.) who remained unequalled in their fields for 1800 years, failed to hit upon it; this, in spite of the fact that Archimedes in his book *On the Number of Sand* (ψαμμίτης) required the use of something like the positional system. We moderns, as a result of continuous custom from childhood, have ceased feeling how profound a scientific achievement is involved.

Besides the principal purpose, let us consider the two additional demands formulated above. In other words, let us determine the most

[2] See, for instance, B. Datta: "Testimony of Early Arab Writers on the Origin of Our Numerals," *Bulletin of the Calcutta Mathematical Society*, vol. 24 (1933).

Other peoples also invented the positional notation independently of the Hindus. Amongst these, besides the Babylonians, were the Mayas of Central America. As early as two thousand years ago they even employed zero as a numeral.

suitable number to be used as a base. There is no unique solution to this problem; the choice partly depends on whether we wish to use our numbers for scientific or for practical purposes. From a purely scientific point of view preference must be given to that number, among the infinite possibilities of choice for a base, which is absolutely distinguished from the rest as the *smallest* among them, namely, the number 2. (Of course, there exists no largest base.) As a matter of fact, in so far as positional notation is employed in purely mathematical investigations, the binary scale ($j = 2$) is regularly chosen. Thus, the number twenty-seven is denoted by:

$$11011 = 1\cdot 2^0 + 1\cdot 2^1 + 0\cdot 2^2 + 1\cdot 2^3 + 1\cdot 2^4 = 1 + 2 + 8 + 16.$$

This example suffices to indicate the inadequacy of the binary system for practical purposes, since it involves undue lengthiness in the representation of large numbers. In practice, therefore, it appears advisable to choose a larger number which shall at the same time have as many factors as possible relative to its size. From this point of view, the numbers 6, 12, 24, 60 suggest themselves, with the first two to be preferred, since in their case the number of primary symbols will remain fairly small. The sexagesimal system ($5\cdot 12$, $2\cdot 12$) which is still employed in the division of days into hours, hours into minutes, and minutes into seconds as well as in the division of the circumference of the circle into 360 degrees, relies on the fair divisibility properties of 60. (The same principle of choosing a number with a relatively large number of factors is to be found in the division of the hour in the Jewish Calendar into 1080 "parts"; for the number $1080 = 2^3\cdot 3^3\cdot 5$ has many factors.) The number 10 upon which our decimal system is based is inferior to 6 or 12, and its predominance is due to the accidental circumstance that man has ten fingers and that primitive man used the fingers for counting.[3]

As late as the Roman period, the Roman numerals, in which the number seventy-eight, for example, appears as LXXVIII, were employed. The example demonstrates the superiority of positional system even in representing relatively small numbers. The use of the alphabet for numerals has an additional drawback which would remain even were the positional principle applied to them, as was proposed by Ibn Ezra (*c.* 1092–1167) with respect to the Hebrew alphabet[4]; for by

[3] For a recent attempt to establish the duodecimal system in common use, see F. E. Andrews: *New Numbers. How Acceptance of a Duodecimal Base Would Simplify Mathematics.* New York, 1935. On the other hand, it has been proposed to calculate by eights instead of tens. See E. M. Tingley in *School Science and Mathematics,* April 1934; cf. *Journal of Educational Research,* January 1937. Cf. *Scripta Math.,* vol. 10 (1944), p. 215.

[4] See the edition of *Sefer Hamispar,* by Rabbi Abraham Ibn Ezra, issued by Moses Silberberg (Frankfurt, 1895). Cf. the essay of A. Loewy: *Über die Zahlbezeichnung in der jüdischen Literatur;* Jeschurun (Berlin), vol. 17 (1930).

assigning definite values to the letters of the alphabet we preclude their use in the representation of undetermined numbers. This would have made impossible the "literal arithmetic" or symbolic algebra which came into use in Europe since the thirteenth century and without which it would be difficult for us to imagine any mathematical calculation.

Let us take as an example the well-known formula

$$(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2.$$

For the Greeks this meant only $(1 + 2)^2 = 1 + 4 + 4$, because α meant 1 and β meant 2. It was thus only a particular arithmetical formula without general significance.[5]

2. The Concept of Cardinal Number

We shall now proceed to construct the numerical concept whose function is to indicate *how many* objects are contained in a given collection or "set." In order to derive this concept as a logical construction which, at the same time, shall retain contact with the psychological aspect of the number concept, we must recognize the fact that the comparison of sets of objects with respect to the number of elements contained in each, is possible without using the concept of number. It is said that there are still some primitive peoples who use for numeration only the three words: "one," "two," "many;" of course, for practical purposes (as in the case of barter) they must also compare sets of objects which contain more than three objects. The means employed for such comparisons is that of establishing a *correspondence* between the objects of one set and those of the other, which is one of the most fundamental and indispensable operations of human thought. By this operation we associate with each element of a set K a unique element of another set L, so that no element of K is paired with more than one element of L. Moreover, in the present case we establish a *one-to-one correspondence* between the two sets by making the additional condition that no two different elements of K shall be paired with the same element in L. If such a correspondence can be established between all the elements of K and L, it is evident that both sets contain the "same number" of elements in the ordinary sense.

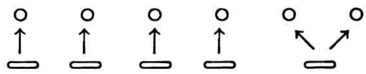
Let us take the example of a barter between a collection K of apples and a collection L of bananas. If the correspondence between these two collections is defined as indicated by the arrows in the following

[5] On the notation of numbers and allied subjects see the popularly written volume by D. E. Smith and J. Ginsburg: *Numbers and Numerals* (Contributions of Mathematics to Civilization, No. 1). New York, 1937.

scheme:

K (Apples):

L (Bananas):



it is a one-valued correspondence of bananas to apples, since to each apple but one banana is related. The last apple, however, is paired with a banana which has already been paired with the preceding apple. As a result, this correspondence is not a one-to-one correspondence (mapping); for were we to change the order of association by reversing the direction of the arrows we would have a banana against which two apples were matched. If, however, we remove the last apple (or for that matter any apple) from the collection, there exists a one-to-one correspondence between the two sets. The existence of such a correspondence indicates that the two sets contain the same number of elements.

From such considerations the concept of number develops by a gradual process of generalization. The first step is to compare various sets of one kind of objects (for example, two collections of apples) with respect to their quantity. In this case we simply match one apple against another without requiring any generalized conceptions. The next step is to match sets whose elements are of different kinds. This requires a more general concept which shall subsume the various kinds of objects under consideration. The necessity for such a step occurs when we wish to compare, for instance, a set of apples with a set of bananas, as done above. In this case the inclusive concept of "fruit" may be formed and we can proceed, by establishing one-to-one correspondences, to compare different collections of fruit. The utmost generalization of such comparison occurs when we are no longer concerned at all with the specific nature of the objects to be matched but merely compare sets of objects or "elements." In making this last step our procedure finally gives rise to the *concept of number*. By abstracting completely from the specific nature of the elements involved we obtain the *common property* of two sets between the elements of which there exists a one-to-one correspondence: the *number of elements* in the set.

In practice, however, we make one further step which may at first appear to be a retrogression. We introduce a universal set, or more precisely, an array of universal sets with which to compare any given set of elements. This universal set lends itself, thus, for use as a common yardstick through the medium of which any different sets may be compared. The elements of these sets have no specific properties: their sole function is to serve as an instrument with which to conduct the process of counting. We may choose them as the sets of the first n natural numbers

$$(1), (1, 2), (1, 2, 3), (1, 2, 3, 4), \ldots, (1, 2, 3, \ldots, n - 1, n).$$

At the present stage of human culture, children learn to employ these universal sets almost at infancy by counting the objects in their vicinity.

If a one-to-one correspondence exists between the elements of two sets they are said to be *equivalent sets*. Accordingly with respect to the relation of equivalence neither set can be distinguished from the other. A one-to-one correspondence between the elements of equivalent sets is also called a "projection," or "mapping," of one set upon the other.

If, on the other hand, in every attempt[6] to establish a one-to-one correspondence between the elements of K and L, there remain elements (at least one element) in one set which *cannot* be paired with an element of the other set, we know without having to resort to the use of numbers that the two sets differ in magnitude. We can even establish an "order of magnitude" between the two sets, which is of practical importance in the example of barter alluded to above; see section 3.

From the idea just described, the concept of number may be derived as follows: To every set K we assign a symbol k to be called "the number of the elements contained in "K" or its *cardinal number*. The cardinal numbers of two sets are equal if, and only if, the sets are equivalent, i.e., if there exists a one-to-one correspondence between their elements. The above construction can be described in a looser form as follows: A set of elements (whether concrete or abstract) may have various properties, such as the specific nature of its elements or the order in which they are arranged. If we ignore all these properties the concept of set is transformed into a more general concept: to each individual element of the set there now corresponds only a "unit" and the set becomes but a collection of units. The new concept formed by way of abstraction will, therefore, be identical for any two sets which are equivalent. This concept we call "the cardinal number of the set."

The method of defining by "way of abstraction" through assigning to a given concept a meaningless symbol as just done in the construction of cardinal numbers, is open to objections from the viewpoint of the general theory of definition. To such objections it may be retorted that we could just as well have singled out a particular set and defined it as

[6] If, as in this section, the number of elements contained in a set is finite, one proves by an arithmetical demonstration (though not a very simple one) that it is not necessary to make several attempts. One attempt with a negative result implies that *any* attempt to establish a one-to-one correspondence will yield a negative result. The property of infinite sets stressed in section 3 involves that the same does not hold with respect to such sets: in comparing infinite sets one has to prove indeed that *any* attempt leads to a negative result, while an individual failure is insignificant.

the cardinal number of all equivalent sets. Thus, we could have designated the set consisting of the sun and the moon as the representative of all pairs; in other words, as the cardinal number 2. For the cardinal number 5, we could have chosen the set of continents. It is obvious that, apart of its inconvenience, such a procedure has to be declined because of its arbitrariness. Nevertheless this method may be employed by using the concept of *ordinal number* (see Chapter II), or by the inductive derivation of numbers in which a cardinal number n is defined as the set of all the numbers $(0, 1, 2, \ldots, n - 1)$ preceding it. For this purpose, among many others, it is convenient to introduce the cardinal number 0 (zero) as the cardinal number of a set without any elements.

Let us return, however, to our analysis of the transition from sets to cardinal numbers. I do not wish to approach this problem from the general philosophical point of view. We should remember that in mathematics the formation of concepts by definition through abstraction is very common. Thus we ascribe to all (oriented) lines parallel to one another a common "direction." Likewise, similar plane figures are said to possess a common geometric "form." All integers which are congruent with respect to a given modulus (see Chapter III) define a common "number-class." All "equivalent" fundamental sequences of rational numbers represent the same real number. Let us clarify the basis for this method of definition and supply its logical justification!

The Greek logicians (especially Aristotle [384–322 B.C.]) as well as those of later periods until two generations ago did not pay sufficient attention to the great differences between the possible predicates which may be associated with various subjects. Their analysis of predication was mistakenly based upon the grammatical forms which propositions assume in language. From the grammatical point of view the propositions "My brothers are stubborn" and "My brothers are similar in appearance," have an identical form. Each of them has a subject and a predicate. Traditional logic regarded all such predicates as qualities. The first of these propositions does in fact associate the quality of stubbornness with each of the brothers. On the other hand, it is evident that the second proposition does not ascribe a quality to any of the brothers. (This fact is very pointedly brought out in the anecdote about a woman who, upon visiting a friend who had just given birth to twins, exclaimed, "How similar your twins are! Especially the one on the right hand.") Such propositions are not concerned with the *qualities* of a single subject but refer to *relations* between two or more subjects which are of equal significance from a logical point of view. (Moreover, a proposition like "I love my children" also expresses a relation between the two subjects "I" and "my children.") We are concerned, therefore, not only with ordinary predication which may be expressed as a propositional function of one variable (e. g., x is beautiful) but with propositional functions of two variables (x resembles y) or of three variables (x is between y and z), etc. A propositional function of one variable is said to be a quality; in the other cases we have a relation of two, three etc., terms.[7]

The most prominent relations in mathematics are those that possess certain very definite properties. In the first place, a relation of two terms may be symmetrical, non-symmetrical, or asymmetrical. Let us denote any relation by the letter R, so that xRy means "x stands in the relation R to y." R is a *symmetrical* relation if, for all values of x and y, xRy implies yRx, that is to say, if the relation is reciprocal. The relations of similarity and parallelism are, thus, symmetrical. R is an *asymmetrical* relation if the truth of xRy implies the falsehood of yRx. Thus, "x is the father of y" or "x is smaller than y" or "x is to the left of y" are asymmetrical relations. A relation which is not symmetrical need not necessarily be asymmetrical. "x is the brother of y" is neither symmetrical nor asymmetrical as the instances, "Moses is the brother of Aaron" on the one hand, and "Moses is the brother of Miriam" on the other, indicate.

A relation R is called *transitive* if xRy and yRz together imply xRz. "x is a descendant of y" and "x is smaller than y" are transitive; but "x is the son of y" is not transitive. Similarly, the relation of similarity in geometry is transitive, but not similarity in its common usage as denoting resemblance. Children are often said to resemble both parents though the parents do not resemble one another.

A relation R which is symmetrical as well as transitive relates any object of its "*field*" to itself (provided that the object stands in the relation to at least one object of the field). In other words, xRx is true for any x of the field. In fact, xRy implies yRx by the symmetry of R, and then xRx follows by the transitivity of R.

The relations that are of greatest importance in mathematics are those which are both symmetrical and transitive, and those which are transitive but asymmetrical. The importance of the second type of relation, "relations of order," will be discussed in section 3. Here we shall be concerned with those relations which possess both symmetry and transitivity, or, as they are called, the *equivalence-relations*. (The word "equivalence" is taken here in a broader sense than that used above.)

[7] The importance of relations is brought out especially in the articles and books of Bertrand Russell. See, e. g., also: *Studies in the Problems of Relations*. University of California Publications in Philosophy, Vol. 13 (1930).

We consider a given equivalence-relation, R, and denote by K a set such that *any two* of its elements, a and b, stand in the relation aRb. In this case any element a of K may be conceived as a "type": the type of a with respect to the relation R. Let us explain this in detail: Every property which, if belonging to a, belongs also to any object related to a by R, is "typical with respect to the relation R." Consequently, any property which is typical with respect to R and belongs to some element of K, belongs also to every other element in K. The set K may, therefore, be regarded as representing properties which are typical with respect to R. We shall consider a few examples.

If by R we refer to the relation of directed parallelism and we define K as the set of all directed straight lines in space parallel to a given line, the common direction of the lines in K is the type of each of them with respect to R. Likewise, by means of the relation of similarity between plane figures, which is also an equivalence-relation, we can derive from a given polygon the concept of a certain polygonal "form." The relation of congruence modulo g (where g is any given natural number above 1) forms from a given integer the concept of a "congruence-class" mod. g.

With this in mind, if we return to the relation of equivalence between sets in the narrow sense described earlier in this section, we see that the concept of cardinal number evolves from a set as its type with respect to the relation of equivalence. For example, the number 5 is the type formed with respect to the relation of equivalence by the set of the fingers of one hand.

It is also possible to define the type as the aggregate of *all* objects related to a given object by the relation R. The cardinal number of a given set is, then, the collection of all sets equivalent to this set. This is the definition of Frege (1848–1925) and Bertrand Russell. Its advantage is that it is formed in accordance with the traditional Aristotelian theory by means of *genus proximum* and *differentia specifica*. On the other hand, it possesses the disadvantage that it involves us in certain logical paradoxes. As a result, Frege despaired of making his theory consistent, whereas Russell, in order to save his definition, set out to construct a new theory of logic: the theory of types.

3. The Ordering of Numbers According to Magnitude

To define the cardinal numbers 1, 2, 3, . . ., we employed the relation of equivalence. To arrange them in their usual order, according to their magnitude, we must employ another relation between sets which is transitive, but unlike equivalence is not symmetrical; in fact, it is asymmetrical. If a set K contains only part of the elements of L, we say that the cardinal number of K is "smaller" than that of L. If we further consider that every set which is equivalent to K has, according to our definition, the same cardinal number as K, we can forego the condition that the elements of K belong actually to L and substitute the following definition:

The (finite) cardinal number k of K is *smaller* than the cardinal number l of L (denoted by $k < l$) if K is equivalent to a proper subset (partial set) of L. (The word "proper" emphasizes the fact that the subset in question contains only part of the elements of L, not all; the emphasis is necessitated by the fact that it is possible, even convenient for certain purposes, to regard every set as a subset of itself.) The same relation between k and l may also be expressed in the form: l is greater than k ($l > k$).

The relation R "x is equivalent to a proper subset of y" is, obviously, transitive and asymmetrical. These two properties are necessary in order to form what in science as well as in common usage is known as an *order-relation*. By means of the relation just defined the numbers may be ordered according to magnitude.

At this point, it should be observed that nowhere in the previous section was it assumed, either explicitly or implicitly, that the sets under consideration contain only a finite number of elements. However, the last definition requires such a limitation. We have made use of the asymmetrical nature of the defined relation R; that is to say, the property that with respect to two sets x and y, xRy and yRx are contradictory. This obviously implies that no set is equivalent to a proper subset of itself or to a proper subset of an equivalent set. This fact known to every child from daily experience, is rigorously proved in arithmetic, but only in so far as finite sets are concerned. The property does not hold when a set contains infinitely many elements, as in the case of the set of all natural numbers 1, 2, 3, . . . This may be shown in the following way.

Let us associate with the set N of all natural numbers the subset M that contains the natural numbers above 1. We establish a one-to-one correspondence by the rule: With every number n of N we associate the number $n + 1$ of M; or in the reverse direction, to every number m of M we relate the number $m - 1$ of N, as illustrated by the following scheme.

$$\begin{array}{lcccccc} N: & 1 & 2 & 3 & \ldots\ldots & n & n+1\ \ldots\ldots \\ & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \updownarrow \\ M: & 2 & 3 & 4 & \ldots\ldots & n+1 & n+2\ \ldots\ldots \end{array}$$

Another and seemingly more concrete example is the following. Let us imagine a sack containing infinitely many oranges, so that to each natural number there corresponds a single orange which is marked with a tag bearing that number. Let us further imagine another sack of the same kind. We take the oranges from the first sack and arrange them in the order of the numbers on their tags, 1, 2, 3, ... From the second sack we take only the oranges marked with even numbers 2, 4, 6, ... and arrange them also in the order of magnitude of the corresponding numbers. We then match orange 1 of the first sack against orange 2 of the second, orange 2 of the first sack against orange 4 of the second, etc. In general, if a is any natural number, we match orange a of the first sack against orange $2a$ of the second. We thus establish a one-to-one correspondence between the oranges of the first sack and "half" of those in the second. The correspondence indicates the equivalence of the set of all *natural* numbers and that of the positive *even* integers. This means that the set of all natural numbers is equivalent to a proper subset of itself which is formed by removing infinitely many of the original elements. In the latter respect this example is even more far-reaching than the previous one.

These examples show that if we take into consideration infinite sets, the relation "x is equivalent to a proper subset of y" is not asymmetrical. The cardinal numbers of infinite sets cannot be ordered, therefore, by this method.