Water Wave Scattering

B.N. Mandal & Soumen De

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B.N. Mandal
Physics and Applied Mathematics Unit
Indian Statistical Institute
Kolkata, India

Soumen De
Department of Applied Mathematics
University of Calcutta
Kolkata, India

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Preface

The theory of water waves is most varied and a fascinating topic. It includes a wide range of natural phenomena in oceans, rivers and lakes. It is mostly concerned with elucidation of some general aspects of wave motion including the prediction of behaviour of waves in the presence of obstacles of some special configurations that are of interest to ocean engineers. Unfortunately, even the apparently simple problems appear to be difficult to tackle mathematically unless some simplified assumptions are made. Fortunately for water, one can assume it to be an incompressible, inviscid and homogeneous fluid. The linearized theory of water waves is based on the assumption that the amplitude of the motion is small compared to the wave length. If irrotational motion is assumed, then the linearized theory of water waves is essentially concerned with solving the Laplace equation in the water region together with linearized boundary condition. There are varied classes of problems which have been/are being studied mathematically in the literature within the frame work of linearized theory of water waves for last many years. Scattering by obstacles of various geometrical configurations is one such class of water wave problems. This book is devoted to advanced mathematical work related to water wave scattering. Emphasis is given on the mathematical and computational techniques required to study these problems mathematically.

The book contains nine chapters. The first chapter is introductory in nature. It includes the basic equations of linearized theory for a single layer fluid, a two-layer fluid, solution of dispersion equations and a general idea on scattering problems and the energy identity in water with a free surface. Chapter 2 is concerned with wave scattering involving thin rigid plates of various geometrical configurations, namely, plane vertical barriers, or curved barriers, inclined barriers, horizontal barriers and also thin elastic vertical plates. For the horizontal case, the barrier is submerged below an ice-cover modelled as a thin elastic plate floating on water. Chapter 3 discusses wave scattering by a rectangular trench by using the Galerkin technique. Chapter 4 involves wave scattering by a dock by using the Carleman singular integral equation followed by reduction to Riemann-Hilbert problems. Chapter 5 involves several wave scattering problems involving discontinuities at the upper surface of water by using the Wiener-Hopf technique, by reduction to the Carleman singular integral equations. Chapter 6 considers scattering by a long horizontal circular cylinder either half immersed or completely submerged. In Chapter 7, some important energy identities are derived for scattering problems in a single-layer and also in a two-layer fluid. Chapter 8 is concerned with wave scattering in a two-layer fluid by a thin vertical plate and by a long horizontal circular cylinder submerged in either of the two layers. Chapter 9 considers a number of wave scattering problems in a single-layer or a two-layer fluid with variable bottom topography by using a simplified perturbation analysis.

It is hoped that this book will be useful to the researchers on water waves. The several wave scattering problems presented in the book are based mostly on the research work carried out by the authors and their associates. The authors thank their young colleagues Harpreet, Rumpa, Dilip and Paramita for their help in the preparation of the manuscript, Prof. S. Banerjea and Prof. U. Basu for their encouragement. They also thank all their associates who worked on problems on water waves.

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B.N. Mandal
Soumen De

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CHAPTER I

Introduction

When a particle in a continuous medium is slightly disturbed from its position of rest then the disturbance is transferred to the neighboring particles and these particles disturb further particles. In this manner the disturbance thus created travels with a definite velocity throughout the medium. This is generally referred to as wave motion. Thus a wave may be regarded as a progressive disturbance propagating from point to point in a continuous medium without displacement of the points. Waves are encountered in almost all branches of mathematical physics such as continuum mechanics, quantum mechanics, acoustics, electromagnetic theory, etc. A wave may be intuitively defined as a recognizable signal that is transferred from one part of a medium to another part, and the signal may be any feature of disturbance which is clearly recognizable and its location at any time can be determined. Ever science waves were studied water waves have served the scientists as models since these can be viewed by the naked eye. Waves are generated due to the existence of some kind of restoring force that tends to bring the system back to its undisturbed state and some kind of inertia that causes the system to overshoot after the system returned to the undisturbed state. One of the common wave motions with which we are most familiar is that of waves occurring at free surface of the liquid with gravity playing the role of the restoring force. These waves are called surface gravity waves. Water waves are such waves. The various wave phenomena in water with a free surface and under gravity have attracted the attention of many famous physicists and mathematician from the eighteenth century. An incomplete list of them includes A.L. Cauchy (1758–1857), S.D. Poisson (1781–1840), J.L. Lagrange (1736–1813), G.B. Airy (1801–1892), G.G. Stokes (1819–1903), Lord Kelvin (Sir William Thompson) (1824–1907), J.H. Michell (1936–1921), H. Lamb (1849–1934), J.J. Stoker (1905–1992), F. Ursell (1923–2012), M.J. Lighthill (1924–1998) and many others. This resulted in a systematic development of the theory of water waves from the latter half of the eighteenth century. This theory has provided a background for somewhat rich development of some important mathematical concepts and techniques and consequently, it has become an important branch of applied mathematics and mathematical physics.

The theory of water waves is the most varied and fascinating subject. It includes a wide range of natural phenomena in oceans, rivers and lakes. The research activities in this area accelerated after the Second World War due to the explosive growth in ocean related industrial activities such as offshore drilling for oil production, construction of offshore structures, extraction of wave energy from ocean waves, design of breakwaters to protect ports, sea resorts and marinas from the rough sea, construction of very large floating structures (floating airports), etc. Interest in the mathematical study of problems on water wave scattering by floating and submerged bodies by applied mathematicians got momentum after the Second World War in UK due to unsuccessful attempts to use portable and floating breakwaters in the surprising amphibious landing by the allied army at Normandy.

In the mathematical study of water wave problems, two types of theories are employed, one is the linearized theory of water waves and the other is the nonlinear shallow water theory. If the wave-length is assumed to be much less than the depth of water, then the effect of disturbance diminishes gradually as one moves downwards away from the surface. Waves in this category are termed as surface waves. In this case if it is assumed that the wave amplitude is small compared to the wavelength, then the corresponding theory is called linearized theory of water waves. Under this theory, it is assumed that the velocity components of the wave potential and the deviation of the upper surface from its mean horizontal position together with their partial derivatives, are small quantities so that their products and powers of higher order can be neglected. Also, various simple assumptions regarding the fluid medium (i.e., water) are made, viz. it is homogeneous, incompressible and inviscid. The motion in water is under the action of gravity only and starts from rest so that it is irrotational. Based on linear theory, any water wave problem can be formulated as a boundary value problem or an initial value problem in which the governing partial differential equation (Laplace equation) is linear and the boundary conditions are linear. Such a formulation is based on the linearized theory of water waves.

A natural question arises about the validity of the linearized theory of water waves from the practical point of view. Ursell et al. (1960) experimented with the height of water waves generated by a flat vertical piston wave maker and obtained results which are in very good agreement with theoretical results predicted under the assumption of linearized theory. Dean and Ursell (1959) and Yu and Ursell (1961) also experimented with a horizontal cylinder in deep water as well as in uniform finite depth water. In both the experiments the experimental results on wave amplitude almost coincide with the theoretical results. These experimental evidences confirm and establish the validity of the linearized theory of water waves. In the present book water wave problems based on only the linearized theory have been considered. Since the last century many ocean technologists have been using the linearized theory to study mathematically various problems related to wave phenomena in ocean and successfully utilize various results predicted under linear theory to ocean related industrial problems.

A different kind of approximation from the foresaid linear theory of waves of small amplitude results when one assumes that the depth of water is sufficiently small compared to some significant length, say the wave length. Then it is not necessary to assume that the deviation of the upper surface of water (free surface) and its slope are small and the resulting theory is no longer a linear theory. There are many circumstances in nature under which such a theory leads to good approximations to various natural phenomena. Among such phenomena are the tides in the oceans, the solitary waves in sufficiently shallow water and the breaking of waves on shallow beaches. The assumption for the shallow water theory leads to a set of nonlinear equations even for the first order approximation and constitutes the theory for the study of long waves. The higher order approximations yield solutions corresponding to continuous permanent finite amplitude waves that can propagate without a change in forms and shapes, known as solitary waves. Again, if the wave amplitude is small so that the velocity components and the free surface elevation (or depression) are small, the nonlinear equations arising in the shallow water theory yield linear hyperbolic partial differential equations. These equations constitute the basis for the theory of tides in the ocean. However, these types of problems will not be considered in this book.

When a train of surface water waves travelling from a large distance is incident on an obstacle submerged or partially immersed in water, some part of the wave is reflected back by the obstacle and some part is transmitted over or below it. This type of problem is known as scattering problem. The reflected and transmitted wave fields involve two constant factors known as the reflection and transmission coefficients respectively. It is supposed that the incident wave field is completely known. However, the resulting wave field, after interacting with the obstacle, is unknown. Determination of the unknown scattered wave field together with the unknown reflection and transmission coefficients constitutes a scattering problem. The two physical quantities, namely the reflection and transmission coefficients are very important since they provide a measure for the amount of reflected and transmitted waves. This information is useful in the construction of offshore structures. Water wave scattering problems have practical importance for various engineering applications. Modeling of breakwaters constructed to protect offshore areas from the impact of a rough sea is one of them. Also, these have applications in designing ships, submarines, offshore structures, etc.

Again, if some part of water surface is covered by some materials such as broken ice (inertial surface) or a rigid plate or an elastic plate, then discontinuities arise in the surface boundary conditions at the regions where the materials meet the surface. Such discontinuities also cause hindrance to an incoming train of surface waves and the phenomena of reflection and transmission of the incident wave train occur in this situation also. For the last few decades there is a considerable interest in the study of various types of water wave problems involving water with a floating ice-cover modeled as a thin elastic plate. The study of these problems has gained considerable importance for quite some time due to two reasons, one is to understand the mechanism and effects of wave propagation through Marginal Ice Zone (MIZ) in polar regions while another is due to their applications in the construction of Very Large Floating Structures (VLFS) like floating oil storage bases, offshore pleasure cities, floating airport, artificial harbors, etc.

In a two-layer fluid with a free surface, waves propagate at two different modes, one along the free surface and the other along the interface between the two layers. The study of wave motion in a two-layer fluid has gained importance due to plans to construct an underwater pipe bridge across Norwegian fjords. A fjord consists of a layer of fresh water on the top of a deep layer of salt water. During winter the fjords are covered by ice, and thus a two-layer fluid where the upper layer has an ice-cover becomes a reality.

We now derive the basic equations of linearized theory of water waves for a single-layer and a two-layer fluid.

1 Basic equations in the linearized theory of water waves

The general hydrodynamic theory is based on the two natural laws governing the fluid motion. These are the laws of conservation of mass and momentum. The basic equations in the linearized theory of water waves are derived from these two laws. It is assumed that water is an inviscid, incompressible and homogeneous fluid and the motion in it is under the only the action of gravity and is irrotational. The assumption on the smallness of motion means that the velocity components together with their derivatives are quantities of the first order of smallness so that their squares, products and higher powers can be neglected. Also the deviation of the upper surface from its mean horizontal position together with its derivatives is assumed to be small. These constitute the basis for the linearized theory. The basic equations in a single-layer fluid (water) and also a two-layer fluid with free surface as well as with ice-cover modeled as thin elastic floating plate are derived below.

1.1 Water with free surface

We consider the motion in water which is assumed to be inviscid, incompressible and homogeneous with constant volume density ρ under the action of gravity g only and bounded above by a free surface. A rectangular Cartesian co-ordinate system is chosen in which the y-axis is taken vertically downwards and the plane $y = 0$ is the position of the undisturbed free surface.

We assume that the motion starts from rest so that it is irrotational and thus can be described by a velocity potential $\Phi(x, y, z; t)$. Hence the fluid velocity $\boldsymbol{q}(u, v, w)$ can be expressed as

$$\boldsymbol{q} = \nabla\Phi. \tag{1.1}$$

Now equation of continuity is

$$\nabla.\boldsymbol{q} = \mathbf{0} \text{ in the fluid region} \tag{1.2}$$

and the Euler's equation of motion in the fluid region is

$$\frac{\partial \boldsymbol{q}}{\partial t} + (\boldsymbol{q}.\nabla)\boldsymbol{q} = \nabla\left(gy - \frac{p}{\rho}\right) \tag{1.3}$$

where p is the fluid pressure.

Using (1.1) the equation of continuity becomes

$$\nabla^2\Phi = 0 \text{ in the fluid region.} \tag{1.4}$$

Integration of (1.3) produces after linearization the linearized Bernoulli's equation

$$\frac{\partial \Phi}{\partial t} = gy - \frac{p}{\rho} \text{ in the fluid region.} \tag{1.5}$$

The pressure p must be equal to the atmospheric pressure at the free surface $y = \eta(x, z, t)$, where η is the free surface depression below the mean horizontal level is

$y = 0$. Now the atmospheric pressure is constant and by a suitable choice of scale it may be taken as zero so that from (1.5) we obtain

$$\frac{\partial \Phi}{\partial t} = g\eta \text{ on } y = \eta(x, z, t). \tag{1.6}$$

This is the dynamical boundary condition at the free surface. Expanding $\frac{\partial \Phi}{\partial t}$ by Taylor's series about $y = 0$ and neglecting higher order terms, the above condition reduces to

$$\frac{\partial \Phi}{\partial t} = g\eta \text{ on } y = 0. \tag{1.7}$$

Again, the linearized kinematic condition at the free surface is

$$\frac{\partial \eta}{\partial t} = \frac{\partial \Phi}{\partial y} \text{ on } y = 0 \tag{1.8}$$

Eliminating η between (1.7) and (1.8) gives the linearized free surface condition as

$$\frac{\partial^2 \Phi}{\partial t^2} = g\frac{\partial \Phi}{\partial y} \text{ on } y = 0. \tag{1.9}$$

The condition of no motion at the bottom gives

$$\nabla \Phi \to 0 \text{ as } y \to \infty \tag{1.10}$$

for deep water and

$$\frac{\partial \Phi}{\partial y} = 0 \text{ on } y = h \tag{1.11}$$

for water of uniform finite depth h.

If the motion is simple harmonic in time with angular frequency ω, then the velocity potential can be expressed as

$$\Phi(x, y, z, t) = \text{Re}\{\phi(x, y, z, t)e^{-i\omega t}\}, \tag{1.12}$$

so that equation (1.4) and the conditions (1.9) to (1.11) become

$$\nabla^2 \phi = 0 \text{ in the fluid region}, \tag{1.13}$$

$$K\phi + \phi_y = 0 \text{ on } y = 0 \tag{1.14}$$

where $K = \frac{\omega^2}{g}$, is called the wave number,

$$\nabla \phi \to 0 \text{ as } y \to \infty \tag{1.15}$$

for deep water,

$$\phi_y = 0 \text{ on } y = h \tag{1.16}$$

for water of uniform finite depth h.

For the two-dimensional case when ϕ is independent of z, the solution of the Laplace equation representing progressive waves is given by

$$\phi(x, y) = \begin{cases} e^{-Ky \pm iKx}, \text{ for deep water,} & (1.17) \\ \dfrac{\cosh k_0 (h-y)}{\cosh k_0 h} e^{\pm ik_0 x}, \text{ for water of depth } h & (1.18) \end{cases}$$

where k_0 is the unique real positive root of the transcendental equation

$$k \tanh kh = K. \tag{1.19}$$

The local solutions are given by

$$\phi = \begin{cases} (k \cos ky - K \sin ky) e^{-k|x|} \ (k > 0), \text{ for deep water,} & (1.20) \\ \dfrac{\cosh k_n (h-y)}{\cosh k_n h} e^{-k_n|x|}, \text{ for water of depth } h, & (1.21) \end{cases}$$

where $\pm ik_n$'s ($n = 1, 2, \cdots$) are the purely imaginary roots of the transcendental equation (1.19). The equation (1.19) is in fact is a relation between the wave number and the angular frequency of a train of surface gravity waves and is known as the dispersion equation. The term is due to the fact that waves whose velocity depends on wave number disperse or separate. Wave dispersion is a fundamental process in many physical phenomena. The dispersion relation (1.9) has roots $\pm k_0$ and $\pm ik_n$ ($n = 1, 2, \cdots, k_1 < k_2 < \cdots$) and there is no other root. These can be easily computed numerically for given K (i.e., given the angular frequency).

1.2 Water with an ice-cover

In this case we consider the motion in water covered by a thin sheet of ice modeled as a thin elastic plate of uniform surface density $\epsilon\rho$, Young's modulus E and Poisson's ratio γ, ϵ being constant having dimension of length, ρ being the density of water. As before we choose the rectangular Cartesian co-ordinate system so that the water (at rest) occupies the region $y \geq 0$, the plane $y = 0$ is the position of the thin ice-cover at rest.

We assume that the motion starts from rest so that it is irrotational and thus can be described by a velocity potential $\Phi(x, y, z, t)$. Then the equation of continuity gives

$$\nabla^2 \Phi = 0 \tag{2.1}$$

in the fluid region and as before the linearized Bernoulli equation is

$$\Phi_t = gy - \frac{p}{\rho}. \tag{2.2}$$

Let $y = \zeta(x, z, t)$ denote the depression of the ice-covered surface below the mean horizontal level. Then Newton's equation of motion for a small element of the ice-covered surface produces Landau and Lifshitz (1959)

$$\epsilon\rho \frac{\partial^2 \zeta}{\partial t^2} = \epsilon\rho g + \Pi - p - L\Delta^4_{(x,z)}\zeta \text{ on } y = 0 \tag{2.3}$$

where Π is the atmospheric pressure and $L = \frac{Eh_0^3}{12(1-\gamma^2)}$ is the flexural rigidity of the thin ice sheet, h_0 being the very small thickness of ice of which still a smaller part is immersed into water and $\Delta^4_{(x,z)}$ representing the two-dimensional bi-harmonic operator. Using (2.2) in (2.3) we get

$$\epsilon\rho \frac{\partial^2 \zeta}{\partial t^2} = \epsilon\rho g + \Pi - \rho\left(g\eta - \frac{\partial \Phi}{\partial t}\right) - L\Delta^4_{(x,z)}\zeta \text{ on } y = 0 \tag{2.4}$$

after linearization. The kinematic condition at the ice-cover is

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \Phi}{\partial y} \text{ on } y = 0. \tag{2.5}$$

Eliminating η between (2.4) and (2.5) we get

$$(\Phi - \epsilon\Phi_y)_{tt} = (1 + D\Delta^4_{(x,z)})g\Phi_y \text{ on } y = 0 \tag{2.6}$$

where $D = \frac{L}{\rho g}$. Also Φ satisfies the bottom condition

$$\nabla\Phi \to 0 \text{ as } y \to \infty \tag{2.7}$$

for deep water and

$$\frac{\partial \Phi}{\partial y} = 0 \text{ on } y = h \tag{2.8}$$

for water of uniform depth h.

Assuming as before the motion to be simple harmonic in time with angular frequency ω, then Φ can be written as

$$\Phi(x, y, z, t) = \text{Re}\{\phi(x, y, z)e^{-i\omega t}\}. \tag{2.9}$$

In this case ice-cover condition (2.6) becomes

$$K\phi + (1 - \epsilon K + D\nabla^4_{(x,y)})\phi_y = 0 \tag{2.10}$$

If the ice-cover is modeled as inertial surface (i.e., non-interacting floating material having no elastic property, e.g., broken ice, floating mat) then $L = 0$ so that $D = 0$. Then the condition (2.10) reduces to

$$K\phi + (1 - \epsilon K)\phi_y = 0 \text{ on } y = 0. \tag{2.11}$$

This takes the form

$$K^*\phi + \phi_y = 0 \text{ on } y = 0, \tag{2.12}$$

if $(1 - \epsilon K) > 0$, where $K^* = \frac{K}{1-\epsilon K} > 0$ and the form

$$K_0\phi - \phi_y = 0 \text{ on } y = 0, \tag{2.13}$$

if $(1 - \epsilon K) < 0$, where $K_0 = \frac{K}{\epsilon K - 1} > 0$. Comparing (2.12) and (2.13) with the usual free surface condition it may be noted that progressive wave along an inertial surface exists if and only if $(1 - \epsilon K) > 0$, i.e., $\omega < \sqrt{g/\epsilon}$. For $(1 - \epsilon K) < 0$, i.e., $\omega > \sqrt{g/\epsilon}$, the condition

(2.13) does not allow existence of any progressive wave at the inertial surface. In this case, the inertial surface is regarded as heavy.

For simplicity let us consider two dimensional motions so that ϕ is a function of x, y only. If we choose $\phi(x,y) = e^{-ky \pm ikx}$, then from the ice-cover condition (2.10) we find that k satisfies the fifty degree polynomial equation

$$D^* k^5 + k - K^* = 0 \tag{2.14}$$

if $(1 - \epsilon K) > 0$, where $D^* = \frac{D}{1-\epsilon K} > 0$,

$$D_0 k^5 + k + K_0 = 0 \tag{2.15}$$

if $(1 - \epsilon K) < 0$, where $D_0 = \frac{D}{\epsilon K - 1} > 0$. It is obvious that the nature of the roots of the polynomial equations (2.14) and (2.15) is the same. Both of them possess unique real positive root. This shows the existence of progressive waves at the ice-cover for any frequency.

For water of uniform finite depth we choose

$$\phi(x, y) = \cosh k(h - y) e^{\pm ikx}$$

then k satisfies the transcendental equation

$$k(D^* k^4 + 1) \sinh kh - K^* \cosh kh = 0 \tag{2.16}$$

for $(1 - \epsilon K) > 0$ and

$$k(D_0 k^4 - 1) \sinh kh - K_0 \cosh kh = 0 \tag{2.17}$$

for $(1 - \epsilon K) < 0$. Each of these two equations (2.16) and (2.17) has a unique real positive root, thus confirming the existence of time-harmonic progressive waves on the ice-cover.

The polynomial equations (2.14) or (2.15) and the transcendental equations (2.16) or (2.17) are the dispersion equations. In section 1.6 we will discuss about the solutions of various dispersion equations in some detail.

For the two-dimensional case, the progressive wave solutions are given by

$$\phi(x, y) = e^{-\lambda_0 y \pm i\lambda_0 x} \tag{2.18}$$

for deep water with ice-cover, where λ_0 is the unique positive root of the polynomial equation

$$Dk^5 + (1 - \epsilon K)k - K = 0 \tag{2.19}$$

and

$$\phi(x, y) = \frac{\cosh \mu_0 (h - y)}{\cosh \mu_0 h} e^{\pm i\mu_0 x} \tag{2.20}$$

for uniform finite depth with an ice-cover, where μ_0 is the unique positive root of the transcendental equation

$$k(Dk^4 + 1 - \epsilon K) \sinh kh - K \cosh kh = 0. \tag{2.21}$$