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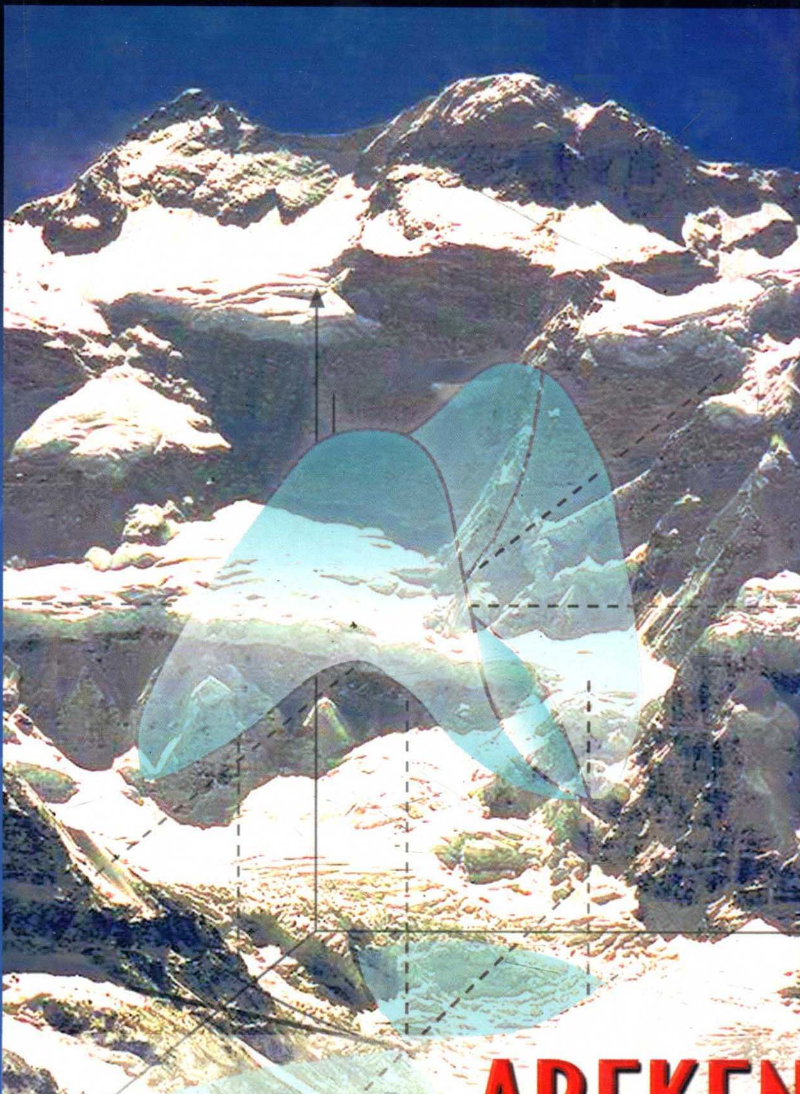
MATHEMATICAL Methods for Physicists



SIXTH EDITION

ARFKEN & WEBER

MATHEMATICAL METHODS FOR PHYSICISTS

SIXTH EDITION

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Vector Identities

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}, \quad A^2 = A_x^2 + A_y^2 + A_z^2, \quad \mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z$$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix}\hat{\mathbf{x}} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix}\hat{\mathbf{y}} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}\hat{\mathbf{z}}$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = C_x\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - C_y\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + C_z\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\,\mathbf{A}\cdot\mathbf{C} - \mathbf{C}\,\mathbf{A}\cdot\mathbf{B}, \quad \sum_k \varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$$

Vector Calculus

$$\mathbf{F} = -\nabla V(r) = -\frac{\mathbf{r}}{r}\frac{dV}{dr} = -\hat{\mathbf{r}}\frac{dV}{dr}, \quad \nabla\cdot(\mathbf{r}f(r)) = 3f(r) + r\frac{df}{dr},$$

$$\nabla\cdot(\mathbf{r}r^{n-1}) = (n+2)r^{n-1}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$$

$$\nabla\cdot(S\mathbf{A}) = \nabla S\cdot\mathbf{A} + S\nabla\cdot\mathbf{A}, \quad \nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0, \quad \nabla\times(S\mathbf{A}) = \nabla S\times\mathbf{A} + S\nabla\times\mathbf{A}, \quad \nabla\times(\mathbf{r}f(r)) = 0,$$

$$\nabla\times\mathbf{r} = 0$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\,\nabla\cdot\mathbf{B} - \mathbf{B}\,\nabla\cdot\mathbf{A} + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B},$$

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

$$\int_V \nabla\cdot\mathbf{B}\,d^3r = \int_S \mathbf{B}\cdot d\mathbf{a}, \quad \text{(Gauss)}, \quad \int_S (\nabla\times\mathbf{A})\cdot d\mathbf{a} = \oint \mathbf{A}\cdot d\mathbf{l}, \quad \text{(Stokes)}$$

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)d^3r = \int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot d\mathbf{a}, \quad \text{(Green)}$$

$$\nabla^2\frac{1}{r} = -4\pi\delta(\mathbf{r}), \quad \delta(ax) = \frac{1}{|a|}\delta(x), \quad \delta(f(x)) = \sum_{i,\, f(x_i)=0,\, f'(x_i)\neq 0} \frac{\delta(x-x_i)}{|f'(x_i)|},$$

$$\delta(t-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-x)}d\omega, \quad \delta(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3}e^{-i\mathbf{k}\cdot\mathbf{r}},$$

$$\delta(x-t) = \sum_{n=0}^{\infty} \varphi_n^*(x)\varphi_n(t)$$

Curved Orthogonal Coordinates

Cylinder Coordinates

$$q_1 = \rho, \quad q_2 = \varphi, \quad q_3 = z; \quad h_1 = h_\rho = 1,\ h_2 = h_\varphi = \rho,\ h_3 = h_z = 1,$$

$$\mathbf{r} = \hat{\mathbf{x}}\rho\cos\varphi + \hat{\mathbf{y}}\rho\sin\varphi + z\hat{\mathbf{z}}$$

Spherical Polar Coordinates

$$q_1 = r,\ q_2 = \theta,\ q_3 = \varphi; h_1 = h_r = 1,\ h_2 = h_\theta = r,\ h_3 = h_\varphi = r\sin\theta,$$

$$\mathbf{r} = \hat{\mathbf{x}}r\sin\theta\cos\varphi + \hat{\mathbf{y}}r\sin\theta\sin\varphi + \hat{\mathbf{z}}r\cos\theta$$

$$d\mathbf{r} = \sum_i h_i dq_i \hat{\mathbf{q}}_i, \quad \mathbf{A} = \sum_i A_i \hat{\mathbf{q}}_i, \quad \mathbf{A}\cdot\mathbf{B} = \sum_i A_i B_i, \mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{q}}_1 & \hat{\mathbf{q}}_2 & \hat{\mathbf{q}}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$\int_V f\, d^3r = \int f(q_1, q_2, q_3) h_1 h_2 h_3\, dq_1\, dq_2\, dq_3 \qquad \int_L \mathbf{F}\cdot d\mathbf{r} = \sum_i \int F_i h_i\, dq_i$$

$$\int_S \mathbf{B}\cdot d\mathbf{a} = \int B_1 h_2 h_3\, dq_2\, dq_3 + \int B_2 h_1 h_3\, dq_1 dq_3 + \int B_3 h_1 h_2\, dq_1 dq_2,$$

$$\nabla V = \sum_i \hat{\mathbf{q}}_i \frac{1}{h_i}\frac{\partial V}{\partial q_i},$$

$$\nabla\cdot\mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(F_1 h_2 h_3) + \frac{\partial}{\partial q_2}(F_2 h_1 h_3) + \frac{\partial}{\partial q_3}(F_3 h_1 h_2)\right]$$

$$\nabla^2 V = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial V}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_2 h_1}{h_3}\frac{\partial V}{\partial q_3}\right)\right]$$

$$\nabla\times\mathbf{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{\mathbf{q}}_1 & h_2\hat{\mathbf{q}}_2 & h_3\hat{\mathbf{q}}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$$

Mathematical Constants

$$e = 2.718281828, \quad \pi = 3.14159265, \quad \ln 10 = 2.302585093,$$

$$1\ \mathrm{rad} = 57.29577951^\circ, \quad 1^\circ = 0.0174532925\ \mathrm{rad},$$

$$\gamma = \lim_{n\to\infty}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1)\right] = 0.577215661901532$$

(Euler-Mascheroni number)

$$B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = B_8 = -\frac{1}{30},\ B_6 = \frac{1}{42}, \ldots \quad \text{(Bernoulli numbers)}$$

$$f(x)=\sum_{n=0}^{\infty} f^{(n)}(a)\frac{(x-a)^n}{n!},\quad f(z)=f(0)+\sum_{n=0}^{\infty} b_n\left(\frac{1}{z-a_n}+\frac{1}{a_n}\right)$$

$$\frac{f'}{f}(z)=\frac{f'}{f}(0)+\sum_{n=0}^{\infty}\left(\frac{1}{z-a_n}+\frac{1}{a_n}\right),\; f(z)=f(0)e^{z\frac{f'}{f}(0)}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{z/a_n}$$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n,\; (1+x)^{\alpha}=\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n,\; e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!},$$

$$\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$$

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!},\quad \cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!},$$

$$\frac{x}{e^x-1}=1-\frac{x}{2}+\sum_{n=1}^{\infty}B_{2n}\frac{x^{2n}}{(2n)!}$$

$$x\cot x=\sum_{n=0}^{\infty}(-1)^n B_{2n}\frac{(2x)^{2n}}{(2n)!},\quad \cot\pi x=\frac{1}{x}+\sum_{n=1}^{\infty}\left(\frac{1}{x-n}+\frac{1}{x+n}\right)$$

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},\quad \zeta(2n)=(-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n}$$

$$\frac{\pi^2}{\sin^2\pi x}=\sum_{n=-\infty}^{\infty}\frac{1}{(x-n)^2},\quad \sin\pi x=\pi x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2}\right)$$

$$\Gamma(z)=\int_0^{\infty}e^{-t}t^{z-1}dt,\quad \frac{1}{\Gamma(x)}=xe^{\gamma x}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-x/n},$$

$$\frac{\Gamma'}{\Gamma}(z+1)=-\gamma+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{z+n}\right)$$

$$e^{iz}=e^{-y}(\cos x+i\sin x),\quad \ln z=\ln|z|+i(\arg z+2\pi n)$$

$$J_\nu(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(\nu+n)!}\left(\frac{x}{2}\right)^{\nu+2n},\quad e^{(x/2)(t-1/t)}=\sum_{n=-\infty}^{\infty}J_n(x)t^n$$

$$(1-2xt+t^2)^{-1/2}=\sum_{l=0}^{\infty}P_l(x)t^l,\; P_l(x)=\frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l(x^2-1)^l,$$

$$\int_{-1}^{1}P_l(x)P_{l'}(x)dx=\frac{2\delta_{ll'}}{2l+1}$$

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 4\pi \sum_{l=0}^{\infty} i^l j_l(kr) \sum_{m=-l}^{l} Y_{lm}^*(\theta_k, \varphi_k) Y_{lm}(\theta, \varphi)$$

$$e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}, \quad H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}), \quad \int_0^{\infty} e^{-x} L_m(x) L_n(x) dx = \delta_{mn}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \; a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx,$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx$$

Integrals

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{i\omega t} dt, \quad f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega) e^{-i\omega t} d\omega$$

$$\int_{-\infty}^{\infty} F(\omega) G^*(\omega) d\omega = \int_{-\infty}^{\infty} f(t) g^*(t) dt,$$

$$\int_{-\infty}^{\infty} g(y) f(x-y) dy = \int_{-\infty}^{\infty} F(\omega) G(\omega) e^{-i\omega x} d\omega$$

$$\int g(z) e^{sf(z)} dz \sim \frac{\sqrt{2\pi}\, g(z_0) e^{sf(z_0)+i\alpha}}{|sf''(z_0)|^{1/2}}, \; f'(z_0) = 0, \; \alpha = \frac{\pi}{2} - \frac{1}{2}\arg f''(z_0)$$

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}, \quad V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')d^3r'}{|\mathbf{r}-\mathbf{r}'|}, \quad \int \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2+m^2}\frac{d^3k}{(2\pi)^3} = \frac{e^{-mr}}{4\pi r}$$

Greek Alphabet

Alpha	A	α	Nu	N	ν
Beta	B	β	Xi	Ξ	ξ
Gamma	Γ	γ	Omicron	O	o
Delta	Δ	δ	Pi	Π	π
Epsilon	E	ϵ	Rho	P	ρ
Zeta	Z	ζ	Sigma	Σ	σ
Eta	H	η	Tau	T	τ
Theta	Θ	θ	Upsilon	Υ	υ
Iota	I	ι	Phi	Φ	ϕ
Kappa	K	κ	Chi	X	χ
Lambda	Λ	λ	Psi	Ψ	ψ
Mu	M	μ	Omega	Ω	ω

MATHEMATICAL METHODS FOR PHYSICISTS

SIXTH EDITION

PREFACE

Through six editions now, *Mathematical Methods for Physicists* has provided all the mathematical methods that aspirings scientists and engineers are likely to encounter as students and beginning researchers. More than enough material is included for a two-semester undergraduate or graduate course.

The book is advanced in the sense that mathematical relations are almost always proven, in addition to being illustrated in terms of examples. These proofs are not what a mathematician would regard as rigorous, but sketch the ideas and emphasize the relations that are essential to the study of physics and related fields. This approach incorporates theorems that are usually not cited under the most general assumptions, but are tailored to the more restricted applications required by physics. For example, Stokes' theorem is usually applied by a physicist to a surface with the tacit understanding that it be simply connected. Such assumptions have been made more explicit.

PROBLEM-SOLVING SKILLS

The book also incorporates a deliberate focus on problem-solving skills. This more advanced level of understanding and active learning is routine in physics courses and requires practice by the reader. Accordingly, extensive problem sets appearing in each chapter form an integral part of the book. They have been carefully reviewed, revised and enlarged for this Sixth Edition.

PATHWAYS THROUGH THE MATERIAL

Undergraduates may be best served if they start by reviewing Chapter 1 according to the level of training of the class. Section 1.2 on the transformation properties of vectors, the cross product, and the invariance of the scalar product under rotations may be postponed until tensor analysis is started, for which these sections form the introduction and serve as examples. They may continue their studies with linear algebra in Chapter 3, then perhaps tensors and symmetries (Chapters 2 and 4), and next real and complex analysis (Chapters 5–7), differential equations (Chapters 9, 10), and special functions (Chapters 11–13).

In general, the core of a graduate one-semester course comprises Chapters 5–10 and 11–13, which deal with real and complex analysis, differential equations, and special functions. Depending on the level of the students in a course, some linear algebra in Chapter 3 (eigenvalues, for example), along with symmetries (group theory in Chapter 4), and tensors (Chapter 2) may be covered as needed or according to taste. Group theory may also be included with differential equations (Chapters 9 and 10). Appropriate relations have been included and are discussed in Chapters 4 and 9.

A two-semester course can treat tensors, group theory, and special functions (Chapters 11–13) more extensively, and add Fourier series (Chapter 14), integral transforms (Chapter 15), integral equations (Chapter 16), and the calculus of variations (Chapter 17).

CHANGES TO THE SIXTH EDITION

Improvements to the Sixth Edition have been made in nearly all chapters adding examples and problems and more derivations of results. Numerous left-over typos caused by scanning into LaTeX, an error-prone process at the rate of many errors per page, have been corrected along with mistakes, such as in the Dirac γ-matrices in Chapter 3. A few chapters have been relocated. The Gamma function is now in Chapter 8 following Chapters 6 and 7 on complex functions in one variable, as it is an application of these methods. Differential equations are now in Chapters 9 and 10. A new chapter on probability has been added, as well as new subsections on differential forms and Mathieu functions in response to persistent demands by readers and students over the years. The new subsections are more advanced and are written in the concise style of the book, thereby raising its level to the graduate level. Many examples have been added, for example in Chapters 1 and 2, that are often used in physics or are standard lore of physics courses. A number of additions have been made in Chapter 3, such as on linear dependence of vectors, dual vector spaces and spectral decomposition of symmetric or Hermitian matrices. A subsection on the diffusion equation emphasizes methods to adapt solutions of partial differential equations to boundary conditions. New formulas have been developed for Hermite polynomials and are included in Chapter 13 that are useful for treating molecular vibrations; they are of interest to the chemical physicists.

ACKNOWLEDGMENTS

We have benefited from the advice and help of many people. Some of the revisions are in response to comments by readers and former students, such as Dr. K. Bodoor and J. Hughes. We are grateful to them and to our Editors Barbara Holland and Tom Singer who organized accuracy checks. We would like to thank in particular Dr. Michael Bozoian and Prof. Frank Harris for their invaluable help with the accuracy checking and Simon Crump, Production Editor, for his expert management of the Sixth Edition.

CONTENTS

CHAPTER 1

VECTOR ANALYSIS

1.1 DEFINITIONS, ELEMENTARY APPROACH

In science and engineering we frequently encounter quantities that have magnitude and magnitude only: mass, time, and temperature. These we label **scalar** quantities, which remain the same no matter what coordinates we use. In contrast, many interesting physical quantities have magnitude and, in addition, an associated direction. This second group includes displacement, velocity, acceleration, force, momentum, and angular momentum. Quantities with magnitude and direction are labeled **vector** quantities. Usually, in elementary treatments, a vector is defined as a quantity having magnitude and direction. To distinguish vectors from scalars, we identify vector quantities with boldface type, that is, **V**.

Our vector may be conveniently represented by an arrow, with length proportional to the magnitude. The direction of the arrow gives the direction of the vector, the positive sense of direction being indicated by the point. In this representation, vector addition

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{1.1}$$

consists in placing the rear end of vector **B** at the point of vector **A**. Vector **C** is then represented by an arrow drawn from the rear of **A** to the point of **B**. This procedure, the triangle law of addition, assigns meaning to Eq. (1.1) and is illustrated in Fig. 1.1. By completing the parallelogram, we see that

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}, \tag{1.2}$$

as shown in Fig. 1.2. In words, vector addition is **commutative**.

For the sum of three vectors

$$\mathbf{D} = \mathbf{A} + \mathbf{B} + \mathbf{C},$$

Fig. 1.3, we may first add **A** and **B**:

$$\mathbf{A} + \mathbf{B} = \mathbf{E}.$$

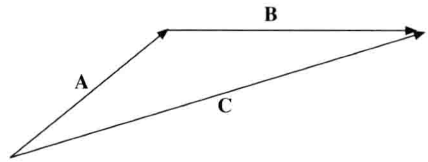

FIGURE 1.1 Triangle law of vector addition.

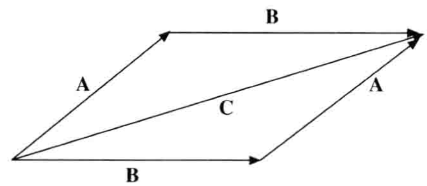

FIGURE 1.2 Parallelogram law of vector addition.

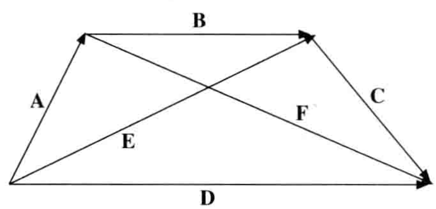

FIGURE 1.3 Vector addition is associative.

Then this sum is added to **C**:

$$\mathbf{D} = \mathbf{E} + \mathbf{C}.$$

Similarly, we may first add **B** and **C**:

$$\mathbf{B} + \mathbf{C} = \mathbf{F}.$$

Then

$$\mathbf{D} = \mathbf{A} + \mathbf{F}.$$

In terms of the original expression,

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}).$$

Vector addition is **associative**.

A direct physical example of the parallelogram addition law is provided by a weight suspended by two cords. If the junction point (O in Fig. 1.4) is in equilibrium, the vector