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COMPLEX ANALYSIS

COMPLEX ANALYSIS

an introduction to the theory of analytic functions of one complex variable

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to the memory of

Ernst Lindelöf

PREFACE

This text is a rigorous introduction on an elementary level to the theory of analytic functions of one complex variable. At American universities it is intended to be used by first-year graduate and advanced undergraduate students.

Since the time the first edition was published there has been a marked change in the quality of American students of mathematics. They enter college better prepared, and they are confronted with true mathematical reasoning at an earlier stage. To a lesser degree the same is true abroad.

In preparing the second edition the author has striven to adjust to this greater maturity of the readers. At the same time the essentially elementary character of the exposition has not been sacrificed. Indeed, nothing could be gained by addressing only the ablest students. Therefore, as in the first edition, the presentation is comparatively broad in the beginning, except for condensed reviews of familiar material, and rises only slowly to a higher level of conciseness. The author has tried to emphasize economy of thought in order to make the reader aware of the intrinsic unity which is so characteristic of the subject.

We enumerate the most important changes from the first edition:

1. The exponential and trigonometric functions are now defined by means of power series. In order to do so it was necessary to introduce an early elementary section on complex power series, a procedure that is not without didactic value in itself.

2. The introduction to point set topology has been rewritten. It now includes the fundamental properties of metric spaces and a more detailed discussion of compactness.

3. Normal families are approached in a more direct manner, and the connection with compactness is emphasized.

4. The Riemann mapping theorem has been combined with a section on the Schwarz-Christoffel formula.

5. A short chapter on elliptic functions has been added. It is deliberately very concentrated in an effort to spare the reader from the customary maze of notations that are needed only by specialists.

6. The exercise sections have been enlarged, and some starred exercises with generous hints for their solution have been included. The latter are to be regarded as part of the text, and students should be encouraged to compose complete proofs.

I should like to take this opportunity to reaffirm my indebtedness to my late teacher Ernst Lindelöf. The whole structure of the book is also deeply influenced by Emil Artin's idea to base elementary homology theory on winding numbers.

I am very grateful to a number of mathematicians who have pointed out errors in the first edition. I can only express a pious hope that no new ones have crept in.

Lars V. Ahlfors

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1 COMPLEX NUMBERS

We want to get to Chapter 4 as quickly as possible, but we will spend some time on Chapters 2 and 3. Quickly through Chapter 1. Our purpose is to prove theorems from known facts about real nos. Pictures will be used only as an aid to imagination.

1. THE ALGEBRA OF COMPLEX NUMBERS

It is fundamental that real and complex numbers obey the same basic laws of arithmetic. We begin our study of complex function theory by stressing and implementing this analogy.

Read very carefully *

1.1. Arithmetic Operations. From elementary algebra the reader is acquainted with the *imaginary unit* i with the property $i^2 = -1$. If the imaginary unit is combined with two real numbers α, β by the processes of addition and multiplication, we obtain a *complex number* $\alpha + i\beta$. α and β are the *real* and *imaginary part* of the complex number. If $\alpha = 0$, the number is said to be *purely imaginary;* if $\beta = 0$, it is of course *real.* Zero is the only number which is at once real and purely imaginary. Two complex numbers are equal if and only if they have the same real part and the same imaginary part.

Addition and multiplication do not lead out from the system of complex numbers. Assuming that the ordinary rules of arithmetic apply to complex numbers we find indeed

$$(\alpha + i\beta) + (\gamma + i\delta) = (\alpha + \gamma) + i(\beta + \delta) \tag{1}$$

and

$$(\alpha + i\beta)(\gamma + i\delta) = (\alpha\gamma - \beta\delta) + i(\alpha\delta + \beta\gamma). \tag{2}$$

In the second identity we have made use of the relation $i^2 = -1$.

It is less obvious that division is also possible. We wish to show that $(\alpha + i\beta)/(\gamma + i\delta)$ is a complex number, provided that $\gamma + i\delta \neq 0$. If the quotient is denoted by $x + iy$, we must have

$$\alpha + i\beta = (\gamma + i\delta)(x + iy).$$

By (2) this condition can be written

$$\alpha + i\beta = (\gamma x - \delta y) + i(\delta x + \gamma y),$$

and we obtain the two equations

$$\begin{aligned} \alpha &= \gamma x - \delta y \\ \beta &= \delta x + \gamma y. \end{aligned}$$

This system of simultaneous linear equations has the unique solution

$$\begin{aligned} x &= \frac{\alpha\gamma + \beta\delta}{\gamma^2 + \delta^2} \\ y &= \frac{\beta\gamma - \alpha\delta}{\gamma^2 + \delta^2}, \end{aligned}$$

for we know that $\gamma^2 + \delta^2$ is not zero. We have thus the result

$$\frac{\alpha + i\beta}{\gamma + i\delta} = \frac{\alpha\gamma + \beta\delta}{\gamma^2 + \delta^2} + i\,\frac{\beta\gamma - \alpha\delta}{\gamma^2 + \delta^2}. \tag{3}$$

Once the existence of the quotient has been proved, its value can be found in a simpler way. If numerator and denominator are multiplied with $\gamma - i\delta$, we find at once

$$\frac{\alpha + i\beta}{\gamma + i\delta} = \frac{(\alpha + i\beta)(\gamma - i\delta)}{(\gamma + i\delta)(\gamma - i\delta)} = \frac{(\alpha\gamma + \beta\delta) + i(\beta\gamma - \alpha\delta)}{\gamma^2 + \delta^2}.$$

As a special case the reciprocal of a complex number $\neq 0$ is given by

$$\frac{1}{\alpha + i\beta} = \frac{\alpha - i\beta}{\alpha^2 + \beta^2}.$$

We note that i^n has only four possible values: $1, i, -1, -i$. They correspond to values of n which divided by 4 leave the remainders 0, 1, 2, 3.

EXERCISES

1. Find the values of

$$(1 + 2i)^3, \qquad \frac{5}{-3 + 4i}, \qquad \left(\frac{2 + i}{3 - 2i}\right)^2, \qquad (1 + i)^n + (1 - i)^n.$$

2. If $z = x + iy$ (x and y real), find the real and imaginary parts of

$$z^4, \quad \frac{1}{z}, \quad \frac{z-1}{z+1}, \quad \frac{1}{z^2}.$$

3. Show that

$$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = 1 \quad \text{and} \quad \left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1$$

for all combinations of signs.

*1.2. **Square Roots.** We shall now show that the square root of a complex number can be found explicitly. If the given number is $\alpha + i\beta$, we are looking for a number $x + iy$ such that

Read very carefully.

$$(x + iy)^2 = \alpha + i\beta.$$

This is equivalent to the system of equations

$$\begin{aligned} x^2 - y^2 &= \alpha \\ 2xy &= \beta. \end{aligned} \tag{4}$$

From these equations we obtain

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = \alpha^2 + \beta^2.$$

Hence we must have

$$x^2 + y^2 = \sqrt{\alpha^2 + \beta^2},$$

where the square root is positive or zero. Together with the first equation (4) we find

$$\begin{aligned} x^2 &= \tfrac{1}{2}(\alpha + \sqrt{\alpha^2 + \beta^2}) \\ y^2 &= \tfrac{1}{2}(-\alpha + \sqrt{\alpha^2 + \beta^2}). \end{aligned} \tag{5}$$

Observe that these quantities are positive or zero regardless of the sign of α.

The equations (5) yield, in general, two opposite values for x and two for y. But these values cannot be combined arbitrarily, for the second equation (4) is not a consequence of (5). We must therefore be careful to select x and y so that their product has the sign of β. This leads to the general solution

$$\sqrt{\alpha + i\beta} = \pm\left(\sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} + i\frac{\beta}{|\beta|}\sqrt{\frac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{2}}\right) \tag{6}$$

provided that $\beta \neq 0$. For $\beta = 0$ the values are $\pm\sqrt{\alpha}$ if $\alpha \geqq 0$, $\pm i\sqrt{-\alpha}$

if $\alpha < 0$. It is understood that all square roots of positive numbers are taken with the positive sign.

We have found that the square root of any complex number exists and has two opposite values. They coincide only if $\alpha + i\beta = 0$. They are real if $\beta = 0$, $\alpha \geqq 0$ and purely imaginary if $\beta = 0$, $\alpha \leqq 0$. In other words, except for zero, only positive numbers have real square roots and only negative numbers have purely imaginary square roots.

Since both square roots are in general complex, it is not possible to distinguish between the positive and negative square root of a complex number. We could of course distinguish between the upper and lower sign in (6), but this distinction is artificial and should be avoided. The correct way is to treat both square roots in a symmetric manner.

EXERCISES

1. Compute

$$\sqrt{i}, \qquad \sqrt{-i}, \qquad \sqrt{1+i}, \qquad \sqrt{\frac{1-i\sqrt{3}}{2}}.$$

2. Find the four values of $\sqrt[4]{-1}$.

3. Compute $\sqrt[4]{i}$ and $\sqrt[4]{-i}$.

4. Solve the quadratic equation

$$z^2 + (\alpha + i\beta)z + \gamma + i\delta = 0.$$

1.3. Justification. So far our approach to complex numbers has been completely uncritical. We have not questioned the existence of a number system in which the equation $x^2 + 1 = 0$ has a solution while all the rules of arithmetic remain in force.

Read very carefully to pg. 6 *

We begin by recalling the characteristic properties of the real-number system which we denote by **R**. In the first place, **R** is a *field*. This means that addition and multiplication are defined, satisfying the *associative, commutative,* and *distributive laws*. The numbers 0 and 1 are neutral elements under addition and multiplication, respectively: $\alpha + 0 = \alpha$, $\alpha \cdot 1 = \alpha$ for all α. Moreover, the equation of subtraction $\beta + x = \alpha$ has always a solution, and the equation of division $\beta x = \alpha$ has a solution whenever $\beta \neq 0$.†

One shows by elementary reasoning that the neutral elements and the results of subtraction and division are unique. Also, every field is an *integral domain:* $\alpha\beta = 0$ if and only if $\alpha = 0$ or $\beta = 0$.

† We assume that the reader has a working knowledge of elementary algebra. Although the above characterization of a field is complete, it obviously does not convey much to a student who is not already at least vaguely familiar with the concept.