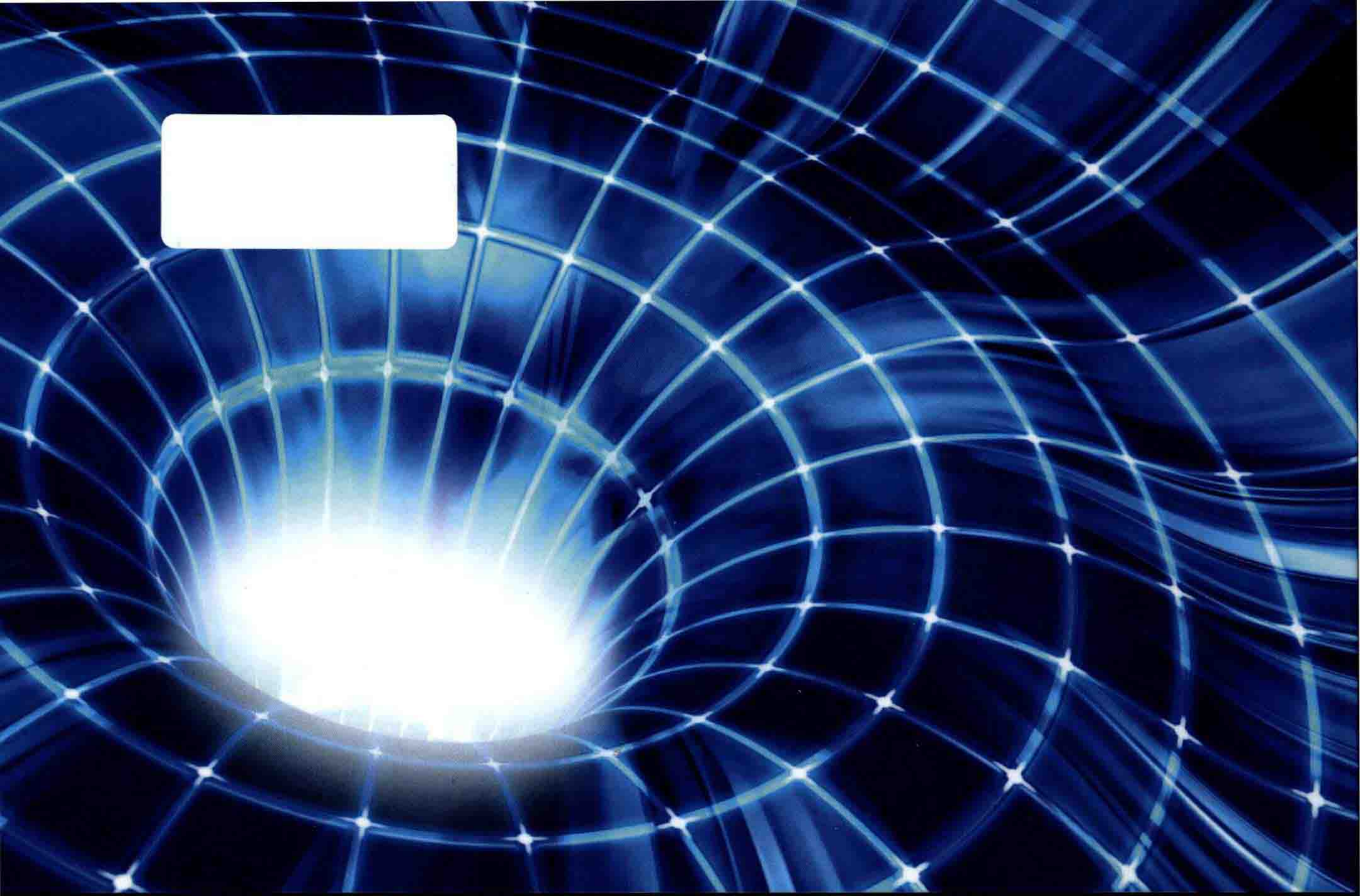

LYAPUNOV EXPONENTS

A Tool to Explore Complex Dynamics

Arkady Pikovsky
and Antonio Politi

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Lyapunov Exponents

A Tool to Explore Complex Dynamics

Lyapunov exponents lie at the heart of chaos theory and are widely used in studies of complex dynamics. Utilising a pragmatic, physical approach, this self-contained book provides a comprehensive description of the concept. Beginning with the basic properties and numerical methods, it then guides readers through to the most recent advances in applications to complex systems. Practical algorithms are thoroughly reviewed and their performance is discussed, while a broad set of examples illustrates the wide range of potential applications. The description of various numerical and analytical techniques for the computation of Lyapunov exponents offers an extensive array of tools for the characterisation of phenomena, such as synchronisation, weak and global chaos in low- and high-dimensional setups, and localisation. This text equips readers with all of the investigative expertise needed to fully explore the dynamical properties of complex systems, making it ideal for both graduate students and experienced researchers.

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Antonio Politi is the 6th Century Chair in Physics of Life Sciences at the University of Aberdeen. He is Associate Editor of *Physical Review E*, a Fellow of the Institute of Physics and of the American Physical Society, and was awarded the Gutzwiller Prize by the Max-Planck Institute for Complex Systems in Dresden and the Humboldt Prize. He is co-author of *Complexity: Hierarchical Structures and Scaling in Physics*.

Preface

With the advent of electronic computers, numerical simulations of dynamical models have become an increasingly appreciated way to study complex and nonlinear systems. This has been accompanied by an evolution of theoretical tools and concepts: some of them, more suitable for a pure mathematical analysis, happened to be less practical for applications; other techniques proved instead very powerful in numerical studies, and their popularity exploded. Lyapunov exponents is a perfect example of a tool that has flourished in the modern computer era, despite having been introduced at the end of the nineteenth century.

The rigorous proof of the existence of well-defined Lyapunov exponents requires subtle assumptions that are often impossible to verify in realistic contexts (analogously to other properties, e.g., ergodicity). On the other hand, the numerical evaluation of the Lyapunov exponents happens to be a relatively simple task; therefore they are widely used in many setups. Moreover, on the basis of the Lyapunov exponent analysis, one can develop novel approaches to explore concepts such as hyperbolicity that previously appeared to be of purely mathematical nature.

In this book we attempt to give a panoramic view of the world of Lyapunov exponents, from their very definition and numerical methods to the details of applications to various complex systems and phenomena. We adopt a pragmatic, physical point of view, avoiding the fine mathematical details. Readers interested in more formal mathematical aspects are encouraged to consult publications such as the recent books by Barreira and Pesin (2007) and Viana (2014).

An important goal for us was to assess the reliability of numerical estimates and to enable a proper interpretation of the results. In particular, it is not advisable to underestimate the numerical difficulties and thereby use the various subroutines as black boxes; it is important to be aware of the existing limits, especially in the application to complex systems.

Although there are very few cases where the Lyapunov exponents can be exactly determined, methods to derive analytic approximate expressions are always welcome, as they help to predict the degree of stability, without the need of actually performing possibly long simulations. That is why, throughout the book, we discuss analytic approaches as well as heuristic methods based more on direct numerical evidence, rather than on rigorous theoretical arguments. We hope that these methods will be used not only for a better understanding of specific dynamical problems, but also as a starting point for the development of more rigorous arguments.

The various techniques and results described in the book started accumulating in the scientific literature during the 1980s. Here we have made an effort to present the main (according to our taste) achievements in a coherent and systematic way, so as to make the understanding by potentially unskilled readers easier. An example is the perturbative approach of the weak-disorder limit that has already been discussed in other reviews; here we present the case of ellyptic, hyperbolic and marginal matrices in a systematic manner.

Although this is a book and, as such, mostly devoted to a coherent presentation of known results, we have also included novel elements, wherever we felt that some gaps had to be filled. This is for instance, the case of the finite-size effects in the Kuramoto model or the extension of the techniques developed by Sompolinsky et al. to a wider class of random processes.

As a result, we are confident that the book can be read at various levels, depending on the needs of the reader. Those interested in the bare application to some simple cases will find the key elements in the first three chapters; the following chapters contain various degrees of in-depth analysis. Cross references among the common points addressed in the various sections should help the reader to navigate across specific items.

The most important acknowledgement goes to the von Humboldt Foundation, which, supporting the visit of Antonio Politi to Potsdam with a generous fellowship, has allowed us to start and eventually complete this project. Otherwise, writing the book would have been simply impossible.

We happened to discuss with, ask and receive suggestions from various colleagues. We specifically wish to acknowledge V. N. Biktashev, M. Cencini, H. Chaté, A. Crisanti, F. Ginelli, H. Kantz, R. Livi, Ya. Pesin, G. Puccioni, K. A. Takeuchi, R. Tonjes and H.-L. Yang.

Antonio Politi wishes also to acknowledge A. Torcini and S. Lepri as long-term collaborators who contributed to the development of some of the results herein summarised.

Special thanks go to P. Grassberger, who, more than 10 years after the publication of a joint paper with G. D'Alessandro, S. Isola and Antonio Politi on the Hénon map, was able to dig out some data to determine the still most accurate estimate of the topological entropy of such a map. As laziness has prevented a dissemination of those results, we made an effort to include them in this book.

We also wish to thank E. Lyapunova, the grand-niece of A. M. Lyapunov, who provided a high-quality photograph of the scientist who originated all of the story.

We finally warmly thank S. Capelin of Cambridge University Press, who has been patient enough to wait for us to complete the work. We hope that the delay has been worthy of a much better product. Although surely far from perfect, at some point we had to stop.

Contents

1 Introduction

1.1 Historical considerations

1.1.1 Early results

The problem of determining the stability of a given regime (e.g. the motion of the solar system) is as old as the concept of the dynamical system itself. As soon as scientists realised that physical processes could be described in terms of mathematical equations, they also understood the importance of assessing the stability of various dynamical regimes. It is thus no surprise that many eminent scientists, such as Euler, Lagrange, Poincaré and Lyapunov (to name a few), engaged themselves in properly defining the concept of stability. Lyapunov exponents are one of the major tools used to assess the (in)stability of a given regime. Within hard sciences, where there is a long-standing tradition of quantitative studies, Lyapunov exponents are naturally used in a large number of fields, such as astronomy, fluid dynamics, control theory, laser physics and chemical reactions. More recently, they started to be used also in disciplines, such as biology and sociology, where nowadays processes can be accurately monitored (e.g. the propagation of electric signals in neural cells and population dynamics).

The reader interested in a fairly accurate historical account of how stability has been progressively defined and quantified can refer to Leine (2010). Here, we limit ourselves to the recapitulation of a few basic facts, starting from the Galilean times, when E. Torricelli (1644) investigated the stability of a mechanical system and conjectured (in the modern language) that a point of minimal potential energy is a point of equilibrium.

Besides mechanical systems, floating bodies provide another environment where stability is naturally important, especially to avoid roll instability of vessels. Unsurprisingly, the first results came from a Flemish (S. Stevin) and a Dutch (Ch. Huygens) scientist: at that time, the cutting-edge technology of ship-building had been developed in the Dutch Republic. In particular, Huygens' approach was quite modern in that he addressed the problem by explicitly comparing two different states. D. Bernoulli too dealt with the problem of roll-stability, emphasising the importance of the restoring forces, which make the body return towards the equilibrium state. L. Euler was the first to distinguish between stable, unstable, and indifferent equilibria and suggested also the possibility of considering infinitely small perturbations.

The concept of stability was further developed by J.-L. Lagrange, who formalised the ideas expressed by Torricelli (for conservative dynamical systems), clarifying that, in the presence of a vanishing kinetic energy, the minimum of the potential energy corresponds to a stable equilibrium. The corresponding theorem is nowadays referred to as "Lagrange-Dirichlet" because of further improvements introduced by J. P. G. L. Dirichlet.

In the nineteenth century, fluid dynamics provided many examples where the stability assessment was far from trivial. Some scientists (notably Lord Kelvin) were striving to unify physics under the paradigm of the motion of perfect liquids, and such an approach required the stability of various forms of motion. At a macroscopic level, in the attempt of predicting the Earth's shape, the problem of determining the stable shape of a rotating fluid, under the influence of the sole action of centrifugal and (internal) gravitational forces, was posed. The studies led to the conclusion that, in some conditions, ellipsoidal shapes are to be expected, but the problem was not fully solved (see Section 1.1.2 on Lyapunov's biography).

On a more microscopic level, hydrodynamics proved to be an extremely fertile field for the appearance of instabilities: concepts such as sensibility to infinitesimal and finite perturbations were present in the minds of esteemed scientists. G. Stokes was one of the pioneers: he stipulated that instabilities naturally occur in the presence of rapidly diverging flow lines, such as past a solid obstacle. Slightly later, H. Helmholtz and W. Thomson discovered that the surface separating two adjacent flows may lose its flatness. Contrary to the instability foreseen by Stokes, which was based only on conjectures, the latter one, nowadays referred to as the Kelvin-Helmholtz instability, was also derived directly from the hydrodynamics equations. Last but not least, Lord Kelvin strived to develop a vortex theory of matter, which, however, required the stability of the underlying dynamical regimes. Only at the end of his career did he convince himself that his ideas were severely undermined by the unavoidable presence of instabilities. The interested reader can look at the exhaustive review by Darrigol (2002).

Celestial mechanics proved to be another fruitful environment for the development of new ideas. In order to appreciate how relevant the subject was in those times, it is sufficient to mention that when P. S. Laplace studied perturbatively the behaviour of three gravitationally interacting particles (the so-called 3-body problem), he referred to it as to the "world system". Heavily relying on recent results by Lagrange, Laplace concluded that the semi-major axis of the orbits is characterised by periodic oscillations. Thus, he concluded in favour of stability, meaning that the fluctuations are bounded. A bit later, S. D. Poisson discovered that second- and third-order terms generate a secular contribution of the type $At \sin \alpha t$; however, as remarked by C. G. J. Jacobi, it was not clear whether such a contribution would survive a higher-order analysis. All in all, no clear answer had yet been given by the end of the nineteenth century. This is the reason why King Oscar II of Sweden decided to offer a prize for those who could find an explicit solution. H. Poincaré won the prize even though he did not actually solve the problem. On the contrary, his work established the existence of unavoidable high sensitivity to initial conditions: what was later called the 'butterfly effect' by the metereologist E. N. Lorenz.[1] Poincaré received the

[1] The expression 'butterfly effect' was arguably introduced by Lorenz in 1972, when he gave a talk at the American Association for the Advancement of Science entitled "Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?"

prize for the revolutionary methods that he developed to gain insight about the behaviour of generic dynamical systems.

A last environment where stability turned out to be of primary importance is related to engineering applications. In the nineteenth century, with the advent of steam engines, it became necessary to regulate the internal pressure inside the boiler. This problem represented the starting point for the birth of a new discipline: automatic control theory. J. C. Maxwell analysed the stability of Watt's flyball regulator by linearising the equations of motion. Independently, I. A. Vyshnegradtsky used a similar approach to study the same problem in greater detail.

1.1.2 Biography of Aleksandr Lyapunov

Here, we briefly summarise some basic facts of the biography of Aleksandr Mikhailovich Lyapunov, mostly relying on Smirnov (1992) and Shcherbakov (1992).

Aleksandr Lyapunov was born in 1857 in Yaroslavl. After completing his gymnasium studies in Nizhny Novgorod, Lyapunov moved to the University of St. Petersburg, where the Mathematical Department was blooming under the direction of Pafnuty Chebyshev, who soon became the supervisor of his graduate studies. Chebyshev used to say that "every young scholar ... should test his strength on some serious theoretical questions presenting known difficulties". As a matter of fact, Lyapunov got involved in a problem that had been earlier proposed to other students (he discovered this later in his career), namely that of determining the shape of a rotating fluid. As his efforts proved unsuccessful, Lyapunov

Fig. 1.1 A. M. Lyapunov in 1902, in Kharkov. Photo courtesy of Elena Alexeevna Lyapunova.

decided to refocus his work, preparing a dissertation entitled *On the stability of elliptic forms of equilibrium of rotating fluids*, which nevertheless allowed him to be awarded a Master's degree in applied mathematics (1884) and made him known in Europe. In 1885, Lyapunov was appointed Privatdozent in Kharkov, where he worked on the stability of mechanical systems. His main results were summarised in a remarkable thesis entitled *The general problem of the stability of motion*, which granted him a PhD at Moscow University (1892). The dissertation contains an extraordinarily deep and general analysis of systems with a finite number of degrees of freedom. Interestingly, Lyapunov mentioned H. Poincaré as one of his principal sources of inspiration.

In 1893, Lyapunov was promoted to ordinary professor in Kharkov. In the following years, he kept studying stability properties of dynamical systems, investigated the Dirchlet problem, and engaged himself in problems of probability theory, contributing to the central limit theorem and paving the way to the rigorous results obtained by his friend Andrei Markov. In 1901 he became head of the department of Applied Mathematics at the Russian Academy of Sciences in St. Petersburg (the position, without teaching duties, had been vacant since 1894, when Chebyshev died).

After having completed a cycle of papers on the stability of motion, Lyapunov came back to the question posed to him by Chebyshev about 20 years before and much related to the problem of determining the form of celestial bodies, earlier formulated by Laplace. While he was still struggling to find a solution, Lyapunov became aware of a book published by Poincaré in 1902 on the same problem and managed to acquire a copy. From a letter sent by Lyapunov to his disciple and close friend Steklov: "To my greatest surprise, I did not find anything significant in this book ... Thus my work has not suffered and I apply myself to it afresh". The book by Poincaré essentially contained previous (known) concepts with little advancements.

Shortly after, the astronomer George Darwin (son of Charles Darwin) published some papers on the same subject, concluding that pear-shaped forms are to be expected. Lyapunov completed his studies in 1905: a treatise of about 1000 pages, with some mathematical calculations made up to 14 digits when necessary. He indeed discovered deviations from ellipsoids, but he also showed that pear-shaped forms are unstable. The controversy with Darwin went on for some years, until it was eventually settled in 1917, when another British astronomer, J. H. Jeans, confirmed that Lyapunov was right.

In 1917 Lyapunov left St. Petersburg for Odessa, so that his wife could receive treatment for tuberculosis. On the day of his wife's death, Aleksandr Lyapunov committed suicide.

1.1.3 Lyapunov's contribution

The first formal definition of stability was given by Lyapunov in his PhD thesis: a given trajectory is stable if, for an arbitrary ε, there always exists a δ such that all other trajectories starting in a δ-neighbourhood of the given one remain at most at a distance ε to it. He introduced also what was later called asymptotic stability, to refer to cases where sufficiently small perturbations eventually die.