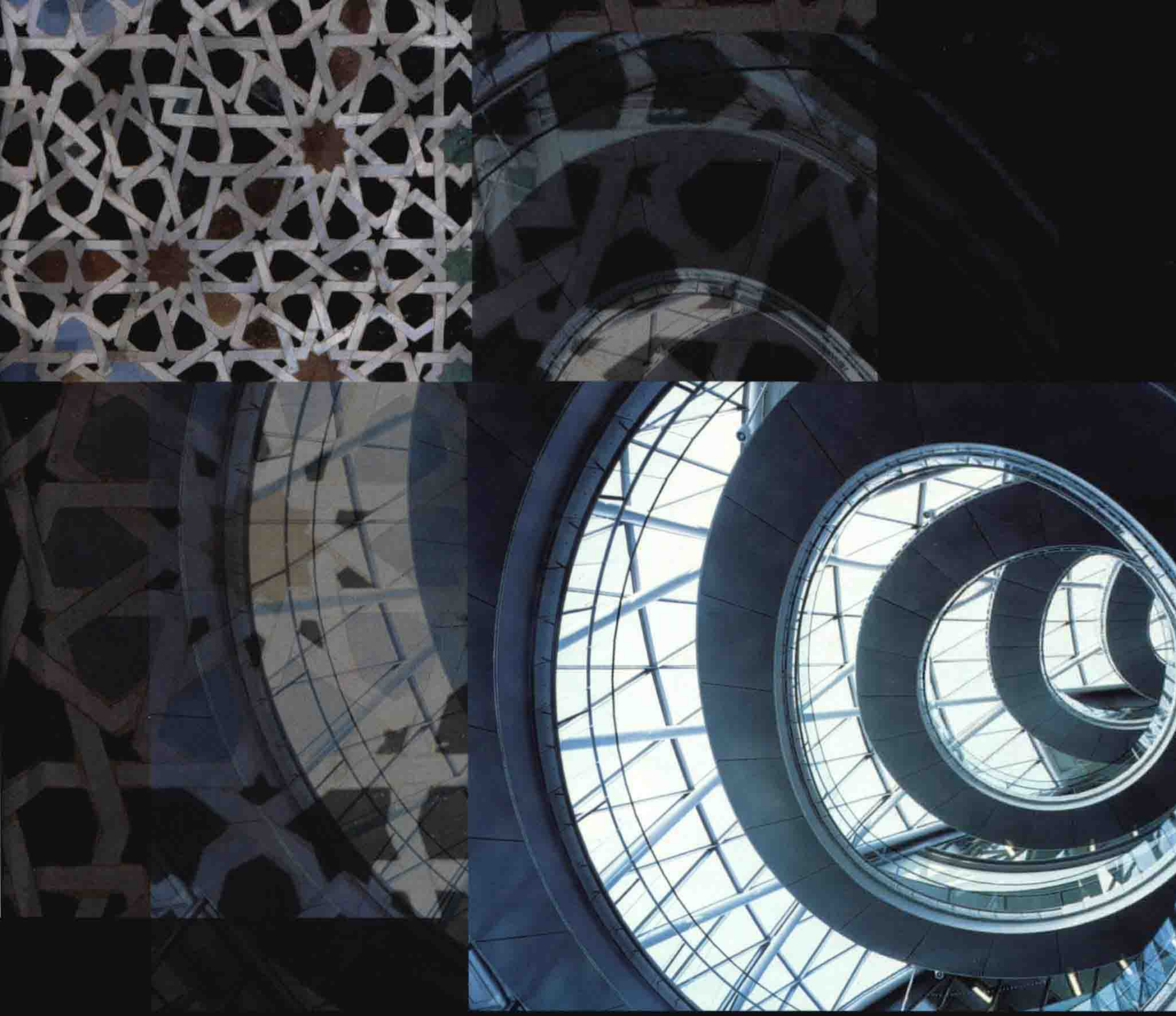

The Changing Shape of Geometry

Celebrating a Century of Geometry and Geometry Teaching

Edited by **Chris Pritchard**

MAA

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The Changing Shape of Geometry

Celebrating a century of geometry and geometry teaching, this book will give the reader an enjoyable insight into all things geometrical. There is a wealth of popular articles including sections on Pythagoras, the golden ratio and recreational geometry. Historical items, drawn principally from the *Mathematical Gazette*, are authored by mathematicians such as G. H. Hardy, Rouse Ball, Thomas Heath and Bertrand Russell as well as some more recent expositors.

Thirty 'Desert Island Theorems' from distinguished mathematicians and educationalists give light to some surprising and beautiful results. Contributors include H. S. M. Coxeter, Michael Atiyah, Tom Apostol, Solomon Golomb, Keith Devlin, Nobel Laureate Leon Lederman, Carlo Séquin, Simon Singh, Christopher Zeeman and Pulitzer Prizewinner Douglas Hofstadter. The book also features the wonderful Eyeball Theorems of Peruvian geometer and web designer, Antonio Gutierrez.

For anyone with an interest in mathematics and mathematics education this book will be an enjoyable and rewarding read.

FRONT COVER PHOTOGRAPHS

Top left: Morocco, Fes, Bou Inaniyya Madrassah. Photographer: Peter Sanders.

Bottom right: Interior view of helical ramp in main atrium, City Hall, London. Photographer: Nigel Young, courtesy of Foster and Partners.

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The Words of Mathematics, by Steven Schwartzman

Foreword

My foreword to this inspiring book unabashedly tells the very personal tale of one individual's various encounters with geometry and geometries over a good number of decades, but the hope behind this act of self-indulgence is that my narration's idiosyncratic aspect is counterbalanced by something more universal in the powerful emotions and the significant experiences recounted.

I grew up deeply in love with mathematics. The lure of graceful patterns and the fascination with the mysterious were simply part of my basic makeup. I first heard about the irrational square root of 2 when I was perhaps eight years old, about imaginary numbers when I was perhaps nine, about matrix multiplication when I was perhaps ten, about Euler's constant e and its connection with i, π, and -1 when I was perhaps eleven, and so forth. All of these things excited me enormously, although, to be sure, I had but a childish grasp of them.

Somehow, though, I did not get much exposure to geometry. I recall that when I was in fourth grade or so, a slightly older friend showed me how to construct a regular pentagon inscribed in a circle with ruler and compass, and for weeks I went around constructing perfect pentagons on every blank piece of paper that came my way, but today, sad to say, I have no more than a blurry memory of how that beloved construction went. As another friend of mine wryly observed a few years ago, "Pentagons are a young man's game."

I also entirely missed out on geometry in school. When I was 13 – the year when I normally would have had a geometry course – my family spent the year in Switzerland, and for various reasons I was exempted from taking mathematics altogether. I recall seeing my classmates work on geometry problems, and being turned off by the way they were forced to acquire the dubious skill of finding absurdly complex 'proofs' of trivially- as well as visually-obvious facts (the two tangents to a circle from an exterior point have the same length, things like that). Oddly enough, I had never encountered the notion of 'proof' in mathematics before, and because I only saw uninspiring examples of the phenomenon, geometry struck me as a set of pedantic exercises, much like counting the number of angles that can dance on the head of a pin (the answer, of course, depends on whether they are acute or obtuse).

After our Genevan year, we returned to America and I started a lifelong habit of avidly browsing the mathematics sections of bookstores. Once in a while I would encounter books with titles like *Advanced Euclidean Geometry* and would flip through them. I vividly recall how I perceived them: as being filled with hugely dense and complicated figures crammed with tightly overlapping circles and lines and far too many letters of the alphabet for my taste. In my young mind, the word 'geometry' took on a stale attic-exuding odor, and the practitioners of the art seemed to be detail-obsessed old fogeys who loved only the most obscure and unimportant nooks and crannies of mathematics.

Something changed a bit when I went to Stanford University. I read about point-line duality and projective geometry in Courant and Robbins' famous tome *What is Mathematics?* as well as in W. W. Sawyer's little gem of a book *Prelude to Mathematics*, and somewhere or other I also ran into Morley's abracadabric manufacture of equilaterality out of nowhere (which I struggled to prove but could not). I encountered the inside-out universes engendered by inversion in a circle when I studied complex variables under Prof. Gordon Latta, and I adored the fact that circles mapped to circles. Perhaps I was even on the verge of falling in love with that musty old game played by musty old fogeys, but then something unfortunate happened.

In those days – the mid-1960s – virtually no courses on geometry were offered at Stanford; however, there was one professor, Harold Bacon, who was known to love geometry, and so I approached him with the idea of doing a reading course with him on projective geometry. I expected to encounter all sorts of magical diagrams like the ones I had seen of Pascal's and Brianchon's theorems, but to my surprise, the book he selected contained very few if any diagrams and seemed to be almost exclusively about linear algebra, with the points of the plane represented by homogeneous coordinates. Although I could appreciate the elegance and symmetry of the matrix equations, I was so disappointed by the abstractness and apictoriality of the course that for many years thereafter I never gave a further thought to geometry.

Convinced nonetheless that I was destined to be a professional mathematician, I started graduate school in mathematics in 1966, but in the courses that I was required to take, I once again encountered that same terrible barrier of enormous abstraction coupled with an utter lack of pictures, and as a result, I became so disillusioned with mathematics that I dropped out, and eventually I wound up doing a doctorate in theoretical physics instead, where I was at least able to keep in touch with certain less forbidding, more concrete areas of mathematics.

One morning in 1992, long after my doctorate and nearly 30 years after my brief brush with geometry at Stanford, I woke up with an image of circles inverting into circles in my mind's eye. How or why this ancient echo of Prof. Latta's amazingly precise and elegant blackboard diagrams flashed into my sleepy head that day, I have no idea, but I remember how much it charmed me and intrigued me, and I decided to prove it for myself that very morning. I tackled the challenge the only way I knew – purely algebraically – and after some work, I managed to get the equations to jump through the hoops I had set up, and lo and behold, the equation of one circle metamorphosed, *mirabile dictu*, into the equation of another circle. Eureka!

I felt so pleased with myself that I asked some mathematically-inclined friends if they, too, could find the proof of this result, and most of them attacked it, as had I, with an algebraic toolkit; one of them, however, showed me a purely synthetic, geometric proof that dazzled me, a proof whose clarity knocked me off my seat. Whereas none of the algebraic proofs offered any insight into WHY one circle's image is another circle, the geometric proof was so direct and so much more to the point that one had the sense of genuinely GRASPING this heretofore elusive fact's *raison d'être*.

This was a beautiful sensation and thus brought me joy, but at the same time it brought me pain, for I felt a sense of shame that I had never experienced this pure brand of clarity before. I realized that for many years, I had confused the reading of any proof whatsoever in a book (or one's own generation of a proof, no matter how jumbly and awkward) with coming to understand WHY, but now I saw that there could be a huge gulf between the two types of experience. Inspired by this new vision, I set out to find or invent sharp, clean, purely geometrical, WHY-revealing proofs for various other geometrical facts that I knew, and within short order, a deep love for geometry – for pure geometry without any algebra – was born in me.

As my initial spark grew into a bonfire, I wanted to share this new passion with others, so I decided to teach a personal seminar on geometry at Indiana University. One day during the first of these 'CaT:DoG' courses ('Circles and Triangles: Diamonds of Geometry'), it occurred to me that I could probably write a computer program to let me construct interrelated circles and triangles and so forth in such a way that I could drag points or lines around on the screen while watching the deep relationships remain invariant. Years earlier, I had seen a computer program like this, in which the images of physical objects could be connected together in intricate ways and then one could move one virtual object on the screen and watch the others react according to the laws of mechanics, so I had a pretty clear image in my head of how my hypothetical geometry program would look. When I mentioned this hope to a very visually-minded friend, he caught me off guard, saying, "I just saw a program exactly like what you've described – I even know the people who designed it, and I'm sure they'd be happy to meet you!" It was in this way that I met the designers of *Geometer's Sketchpad* (who kindly gave me a free copy of their product!), and I was thereby spared the effort of writing such a program myself (or getting a student to do it for me).

Over the next few years, I delved into many areas of geometry, making quite a few discoveries of my own and encouraging my students to do the same. Crystal-clarity of proofs became a religion for me, and I was ever in search of simpler and simpler ways of understanding facts for which I already had a decent, perhaps simple, proof. Moreover, the fascinating way in which the use of analogy and the extrapolation of patterns allow mathematicians to generate fresh new concepts and to reperceive and reconceive familiar ideas and patterns in radically new ways became central to my personal thinking and to my CaT:DoG courses. (In this volume, there are many stunning examples of this deep phenomenon of flexible reperception and analogy-based creativity, but I personally was struck by two lovely variations on the theme of Pythagoras – one in the article by Hazel Perfect, the other in an article by Larry Hoehn.)

As my experiences with geometry grew deeper and wider, I started exploring not just geometry but geometries, and my goal of understanding, in as clear a manner as possible, the interconnections between different sorts of geometries became another passion. For instance, I knew very well that deleting one line from the projective plane leads to Euclidean geometry, and so one day, when in a playfully dualistic mood I asked myself what would happen if instead I were to delete a point from the projective plane, I was rapidly and inevitably led to a very disorienting new geometry that I dubbed 'Euclidual' (meaning 'dual to Euclidean' and rhyming with 'residual'), in which lines are closed and have finite length, and points, too, have bizarre new counterintuitive properties, including the existence of 'parallel points'. I investigated the shape of 'circuals' in this new geometry, I found that the sum of the three 'internal slides' of a 'trislide' is π, and so forth.

I soon discovered that deleting not just one but a whole family of lines from the projective plane leads to the non-Euclidean geometry discovered by Janos Bolyai and Nikolai Lobachevsky, and that the dual act of deleting a family of points instead leads to what I naturally dubbed 'non-Euclidual geometry'. And then, mirroring the marvelous self-duality of projective geometry, there were even new self-dual geometries obtained by simultaneously deleting a line plus a point, or else a family of lines plus a family of points. Altogether, then, I wound up discovering an elegant family of exactly nine different two-dimensional geometries obtained by various combinations of simple deletions from the projective plane. I then had the chutzpah to write to one of my heroes, Donald Coxeter, and ask him if any of this was new.

Coxeter's reply to me, extremely cordial and gracious, led me to the realization, after some bibliographical research, that my personal 'new' geometries had in fact been found by Arthur Cayley over one hundred years earlier – a fact that was rather a crushing blow. A year or two later, on a visit to Toronto where I met Coxeter in person, I also made the acquaintance of Abe Shenitzer, who gave me a wonderful book on non-Euclidean geometry by the Russian mathematician I. M. Yaglom (Shenitzer had translated it into English), in which I found ‘my own’ 3×3 family of plane geometries explored in an extraordinarily systematic and incisive fashion – another blow to my ego. My only solace lay in the thought that I had come across these geometries via a different and somewhat more down-to-earth route.

Since then, despite occasional setbacks in which I have realized that some exquisite recent ‘discovery’ that I think of as mine alone is in fact someone else’s rather old hat, I have continued my avid explorations of Euclidean geometry with the aid of *Geometer’s Sketchpad*, not to mention the aid of many students and friends. I can safely say that I have developed as profound a love for geometry as I have ever had for any branch of mathematics. But how did my perspective on it change so radically? After all, those old books on ‘advanced Euclidean geometry’ that once struck me as infinitely dull and musty trunks in the attic now strike me as treasure boxes filled with deeply intoxicating magic.

All I can say is that geometry, like fine wine, is an acquired taste. Every theorem and every idea has its own unique bouquet. Some appeal to certain individuals, others to other individuals. There are fine French wines, fine Italian wines, fine German wines, and so forth. Poncelet, Gergonne, Brianchon, Beltrami, Klein, Plücker, von Staudt, Feuerbach ... The list goes on virtually forever, and I must admit that I occasionally fantasize about how such old-time geometers would be thrilled if they could only watch over my shoulders as I drag a point about on my laptop’s screen and reveal such magical phenomena as the fourfold kiss of Feuerbach’s theorem or the uncanny precision of Poncelet’s porism. But such things are not to be. The dead are dead, and they will never know that particular joy, though I like to think that in some sense they knew these phenomena so intimately inside their heads that they might not even have been surprised to witness them taking place in dynamic color before their eyes.

Each generation is a product of its age and its culture, and ours is of course a computer culture. We few who love geometry are privileged to be able to witness great depth and visual magic on our screens – and yet, how are computers used by most school-aged children? Mostly for playing video games and watching movies – only very rarely for learning mathematics, let alone exploring it on their own. It’s the same story as for television, just retold an octave higher: a marvelous technology is invaded and possessed by the money-driven entertainment industry, and what once was the most promising of educational tools becomes instead a bottomless pit of banality and triviality.

And yet there is still hope. With Euclidean and even non-Euclidean geometry now directly visualizable on screens, with visual treatments of four-dimensional special relativity and even of Riemannian manifolds now accessible with a few flicks of a cursor, high school and university students are going to be able to penetrate rapidly into conceptual universes that previous generations could barely imagine. In my opinion, thanks to these new sorts of visualization tools, geometry is poised on the threshold of a new golden era.

But what is this thing called 'geometry'? As Michael Atiyah suggests in this volume, it may be that geometry is less a specific branch of mathematics than a way of looking at any or all branches of mathematics – it may be that geometry is simply that which one does when one visualizes, as opposed to what one does when one calculates or cogitates. And it is certainly undeniable that there has been a great turnaround since the incredibly depressing picture-barren days when I was an undergraduate mathematics student. In those days, whole textbooks on such undeniably geometrical topics as topology were standardly written with nary a figure anywhere. Today, austerity that grim would be unthinkable. Indeed, just a few years ago, a book with the title *Visual Complex Analysis* (by Tristan Needham) garnered high prizes for its intuitive manner of presenting ideas that formerly were presented in texts that scorned intuition, that revered rigor and arcane symbolism, and that were totally or almost totally devoid of visual explanations. By contrast, Needham's work of art, with its hundreds and hundreds of beautiful figures *à la* Latta, brings complex analysis alive in an unprecedented manner.

Along these same lines, a horribly opaque, utterly algebraic, mind-numbingly subscript-studded proof of Van der Waerden's exquisite theorem ('If you color every positive integer either red or green, then no matter how you do it, one or the other color will contain arithmetic progressions of arbitrary length'), such as is given by Alexander Khinchin in his poetically titled but antipoetically written little book *Three Pearls of Number Theory*, is now, thank God, extremely outmoded, having yielded its place to a highly diagrammatic and transparently intuitive proof such as can be found in *Ramsey Theory* by Ronald Graham *et al.* Though Van der Waerden's theorem is not usually thought of as geometry, such a highly visual proof would bring it under that rubric, at least according to Atiyah's characterization.

There is an ironic aspect of Atiyah's characterization, though, which is the fact that much of what has traditionally been labeled 'geometry' would not count as geometry, as it is usually presented in an unapologetically algebraic and strictly apictorial manner; conversely, areas of mathematics that traditionally have never been considered 'geometry' could acquire that label as they are rendered more and more pictorial and accessible to the mind's eye.

This recent movement toward the invitingly visual and away from the forbiddingly formal is a marvelous trend, and I hope that it continues to flourish. I must nonetheless admit that when I flip through contemporary schoolbooks that purport to teach geometry, I find an appalling lack of rigor and an appalling lack of visual beauty. To be sure, I find plenty of bright colors on every page, plenty of different typefaces and distracting typographical virtuosity, and plenty of allusions to the worlds of advertising and clothes and engineering and space travel and sports, but almost no links whatsoever to the world of – well, the world of mathematics.

It would be in vain that one would search today's high school geometry texts for such beautiful concepts as the Euler line, the nine-point circle, Feuerbach's theorem, Desargues' theorem, Pappus' theorem, Pascal's theorem, Ceva's theorem, Menelaus' theorem, inversion in a circle, and on and on and on. Where did it all go? Why were such things commonplace in texts 80 years or so ago, whereas now they are totally unheard of? Why was the initially austere and perhaps pointless-seeming but subsequently deeply rewarding activity of erecting an intellectual edifice by means of logical deduction taken for granted as part of school curricula back then? Why does mathematics today have to be 'relevant' and 'fun', have to prove its worth by chewing gum like a sports star, acting sexy like a movie star, spouting cutesy sound-bites like a with-it journalist, displaying itself as eye candy like a top model – but, heaven forbid, not by exploring unsuspected symmetries and subtle patterns purely for their own sake, like a scientist (let alone a mathematician!)?

I have no final answers to offer, ultimately, to these conundra. I am baffled and troubled by the widespread societal loss of interest in and respect for the beauty of pure mathematics. All I can say is that each age has its pluses and its minuses, its own unique points of view and its own unique blind spots. Today we find ourselves somewhere along a vast swing of a vast pendulum, and its arc may not be a periodic one. Who knows where the swing of mathematics education will end up? All we can do is try our best to influence it in the manner we feel is most useful and most beautiful – and it is indubitably in the spirit of high reverence for hidden beauty that Chris Pritchard has fashioned this book, his contribution to the ongoing debate.

May this book have a profound effect on the trajectory of the swinging pendulum! May geometry soon enjoy a grand reflowering! May our children come to appreciate the subtle and sublime harmonies that link triangles and quadrangles, conic sections and cross ratios, polygons and polyhedra, tessellations and tori, curvatures and conjugacies, poles, polars, and polarities! Long live the ancient art and the timeless poetry of geometry!

Douglas Hofstadter
Center for Research on Concepts and Cognition
Indiana University

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