RANDOM MEASURES

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Department of Mathematics
Chalmers University of Technology and University of Göteborg

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Preface

My aim in writing this book has been to give a systematic account of those parts of random measure theory which do not require any particular order or metric structure of the state space. The main applications are of course to random measures on Euclidean spaces, but since most proofs apply without changes to the case of arbitrary locally compact second countable Hausdorff spaces, I have chosen to work throughout within this more general framework.

By a random measure on a topological space $\mathfrak{S}$ is meant a measurable mapping from some abstract probability space into the space $\mathfrak{M}$ of locally finite measures μ on $\mathfrak{S}$, where the σ-field in $\mathfrak{M}$ is taken to be the one generated by the mappings $\mu \to \mu B$ for arbitrary Borel sets B in $\mathfrak{S}$. The most convenient way of treating simple point processes on $\mathfrak{S}$, i.e. locally finite random subsets of $\mathfrak{S}$, is in terms of the induced counting random measures, which motivates the study of integer valued random measures. There are strong reasons, however, for widening the scope of study beyond the integer valued case:

First of all, general random measures are indispensable within point process theory itself. Thus they are needed to describe the asymptotic behavior of thinnings, and they further play a basic role in the context of conditioning. Secondly, many useful ideas and techniques of point process theory are equally important in the case of arbitrary random measures, and many results have interesting extensions to the general case. In particular, there exists an interesting analogy between certain results for simple point processes and for diffuse random measures, respectively. Thus general random measures appears to be the natural scope of a theory. A third reason is the fact that general random measure theory is rapidly developing into a universal tool for dealing with problems in possibly unrelated areas of probability, such as exchangeability, regenerative sets, stochastic geometry, and particle systems, just to mention a few.

Due to the late development of the subject, the basic ideas of random measure theory have not yet found their way into the standard textbook literature. In order to develop an intuitive feeling for the subject, it may therefore be helpful for newcomers in the field to start with some less demanding introductory text, like those by Cox/Isham (1980), Grandell (1977) and Neveu (1977). As with any other highly developed branch of modern probability, a deeper understanding of random measure theory requires a thorough knowledge of probabilistic measure theory, at the level of e.g. Bauer (1972), Bellach et al. (1978), Billingsley (1979) or Neveu (1969). For the benefit of the reader, some general facts from topology, measure theory and probability have been collected in an appendix.

The book reflects my own research in the area, both in spirit and in scope, and there is very little overlap with other books in the field. (The monograph by Matthes/Kerstan/Mecke (1978) is strongly recommended for supplementary reading!) Many classical results are presented here in extended form and with simple unified proofs based on modern analytic, topological and measure theoretic arguments. Several results appear in print for the first time. This applies especially to the conditioning theory presented in the last three chapters, which was developed exlusively for the present edition. (For this new edition, I have also made some additions and improvements in previous chapters.) The exercises at the end of each chapter provide complements to and extensions of results in the main text.

Let me finally acknowledge my gratitude and debt to all those, whose interest, encouragement and criticism have influenced my work in various ways. It all started in the fall of 1971, when Peter Jagers raised my interest in random measures by organizing a seminar and a mini-conference on the subject, here in Göteborg. The first edition of this book was based on my lectures during the spring semester of 1974 at the Statistics Department, University of North Carolina at Chapel Hill, where I enjoyed great hospitality and benefitted from many valuable comments of the audience. I am especially grateful to Ross Leadbetter for his constant encouragement and detailed criticism. Through the mail, I also enjoyed the stimulating interest and valuable advice of Prof. K. Matthes. Since the appearance of the first edition, I have benefitted from enumerable conversations on related matters, especially with Jan Grandell and with scientists from GDR. Alan Karr kindly provided a long list of comments to the first edition, whereas Pieter van der Hoeven gave me access to his as yet unpublished work and helped me to understand its subtleties.

On the personal level, these have been years of disasters and humiliation. Luckily Erik and Anki, my beloved children, provided moments of true happiness. My heartfelt love goes to them.

Göteborg in May 1982

Olav Kallenberg

Contents

1. Foundations

1.1. The basic spaces

Let $\mathfrak{S}$ be a fixed locally compact second countable Hausdorff topological space, (cf. 15.6 in the Appendix). Such a space is known to be *Polish*, i.e. there exists some separable and complete metric ϱ in $\mathfrak{S}$ generating the topology. Though we shall often make use of such metrizations, it should be noticed that our definitions and results never depend on any particular choice of ϱ.

In $\mathfrak{S}$ we introduce the Borel algebra $\mathcal{S}$, i.e. the σ-algebra generated by the topology, and further the ring $\mathcal{B}$ consisting of all *bounded* (i.e. relatively compact) sets in $\mathcal{S}$. We write $\mathcal{F} = \mathcal{F}(\mathfrak{S})$ for the class of all $\mathcal{S}$-measurable functions $f: \mathfrak{S} \to \mathbb{R}_+ = [0, \infty)$ (a notation to be retained for more general spaces $\mathfrak{S}$), and we let $\mathcal{F}_c = \mathcal{F}_c(\mathfrak{S})$ denote the subclass of all functions in $\mathcal{F}$ which are continuous with compact support.

To simplify statements, we introduce three types of subclasses of $\mathcal{B}$. By a *DC-ring* (D for dissecting, C for covering) we shall mean a ring $\mathcal{U} \subset \mathcal{B}$ with the property that, given any $B \in \mathcal{B}$ and any $\varepsilon > 0$, there exists some finite cover of B by $\mathcal{U}$-sets of diameter less than ε (in any fixed metrization of $\mathfrak{S}$). A *DC-semiring* is a semiring $\mathcal{I} \subset \mathcal{B}$ with the same property. (Recall that a semiring is a class $\mathcal{I}$ of sets which is closed under finite intersections and such that any proper difference between $\mathcal{I}$-sets may be written as a finite disjoint union of sets in $\mathcal{I}$.) Note that DC-rings and DC-semirings exist and may even be chosen countable. (Indeed, the ring generated by a countable base formed by bounded sets is a countable DC-ring.) On the line $\mathbb{R}$, typical DC-semirings and DC-rings are families of intervals and interval unions respectively, hence our notations $\mathcal{I}$ and $\mathcal{U}$. We shall finally say that a class $\mathcal{C} \subset \mathcal{B}$ is *covering* if every set $B \in \mathcal{B}$ has a finite cover of $\mathcal{C}$-sets.

Lemma 1.1. *The notions of DC-semiring and DC-ring are independent of the choice of metric ϱ in $\mathfrak{S}$.*

Proof. Consider two metrizations ϱ and ϱ' of $\mathfrak{S}$, and suppose that $\mathcal{I} \subset \mathcal{B}$ is a DC-semiring (or DC-ring) w.r.t. ϱ'. To prove the DC-property w.r.t. ϱ, let $B \in \mathcal{B}$ and $\varepsilon > 0$ be arbitrary. By 15.6.1, we may choose an open bounded set $G \supset B^-$ (B^- denoting the closure of B), and since G^- is compact, the identity mapping of G onto itself is uniformly continuous w.r.t. ϱ' and ϱ. It follows in particular that there exists some $\varepsilon' > 0$ such that $\varrho(s, t) < \varepsilon$ whenever $s, t \in G$ with $\varrho'(s, t) < \varepsilon'$. From the compactness of B^-, it is further seen that $\varrho'(B^-, G^c) > 0$ (G^c denoting the complement of G), and so we may assume that $\varepsilon' < \varrho'(B^-, G^c)$. Now suppose that $\mathcal{I}' \subset \mathcal{I}$ is a finite cover of B by $\mathcal{I}$-sets with ϱ'-diameters $< \varepsilon'$. Since we may discard any set whose intersection with B is empty, we may assume that the covering sets are all subsets of G. But then their ϱ-diameters must be less than ε. □

For any class $\mathcal{C} \subset \mathcal{B}$, let $\hat{\sigma}(\mathcal{C})$ denote the smallest ring which contains $\mathcal{C}$ and is closed under bounded countable unions.

Lemma 1.2. *Let $\mathcal{I} \subset \mathcal{B}$ be a DC-semiring. Then $\hat{\sigma}(\mathcal{I}) = \mathcal{B}$.*

Proof. For any compact set $C \subset \mathfrak{S}$, let $G_1, G_2, \ldots$ be bounded open sets satisfying $G_n \downarrow C$, (cf. 15.6.1 for existence), and choose for each $n \in \mathbb{N} = \{1, 2, \ldots\}$ a finite cover $\{I_{nj}\} \subset \mathcal{I}$ of C which is contained in G_n, (cf. the proof of Lemma 1.1). Then $C = \bigcap_n \bigcup_j I_{nj}$, so we get $C \in \hat{\sigma}(\mathcal{I})$. For a fixed compact $C \subset \mathfrak{S}$, we now define

$$\mathcal{D} = \{B \in \mathcal{S}: B \cap C \in \hat{\sigma}(\mathcal{I})\},$$

and note that $\mathcal{D}$ contains $\mathfrak{S}$ and is closed under proper differences and monotone limits. Furthermore, it was shown above that $\mathcal{D}$ contains the class $\mathcal{C}$ of all compact sets, and $\mathcal{C}$ being closed under finite intersections, it follows from 15.2.1 that $\mathcal{D}$ contains the σ-algebra $\sigma(\mathcal{C})$ generated by $\mathcal{C}$. Since $\sigma(\mathcal{C}) = \mathcal{S}$, this proves that $\mathcal{B} \subset \hat{\sigma}(\mathcal{I})$. Conversely, $\mathcal{B}$ is clearly closed under bounded countable unions, so $\mathcal{B} \supset \hat{\sigma}(\mathcal{I})$. □

Say that a measure μ on $(\mathfrak{S}, \mathcal{B})$ is *locally finite* (or *Radon*) if $\mu B < \infty$ for all $B \in \mathcal{B}$, and let $\mathfrak{M} = \mathfrak{M}(\mathfrak{S})$ denote the class of all such measures. We further introduce the subclass $\mathfrak{N} = \mathfrak{N}(\mathfrak{S})$ of all measures $\mu \in \mathfrak{M}$ with $\mu B \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $B \in \mathcal{B}$. Let $\mathcal{M} = \mathcal{M}(\mathfrak{S})$ and $\mathcal{N} = \mathcal{N}(\mathfrak{S})$ be the σ-algebras in $\mathfrak{M}$ and $\mathfrak{N}$ respectively *generated* by the mappings $\mu \to \mu B$, $B \in \mathcal{B}$, (i.e. the smallest σ-algebras making these mappings measurable).

Given any $\mu \in \mathfrak{M}$ and $f \in \mathcal{F}$, we define the integral μf and the measure $f\mu$ by

$$\mu f = \int_{\mathfrak{S}} f(s)\, \mu(ds); \quad (f\mu)\, B = \int_B f(s)\, \mu(ds), \quad B \in \mathcal{B}.$$

In the particular case when f equals the *indicator* 1_B of B (being equal to 1 on B and to 0 elsewhere) for some $B \in \mathcal{S}$, the measure $f\mu = 1_B\mu$ is called the *restriction* of μ to B, and we write for brevity $1_B\mu = B\mu$. The notations introduced above will also be used for measures on more general spaces than $\mathfrak{S}$.

Lemma 1.3. *The mapping $\mu \to \mu f$ of $(\mathfrak{M}, \mathcal{M})$ or $(\mathfrak{N}, \mathcal{N})$ into $[0, \infty]$ is measurable for every $f \in \mathcal{F}$. If $f \in \mathcal{F}$ is bounded, then $\mu \to f\mu$ is a measurable mapping of $(\mathfrak{M}, \mathcal{M})$ or $(\mathfrak{N}, \mathcal{N})$ into $(\mathfrak{M}, \mathcal{M})$, and in particular, $\mu \to B\mu$ is a measurable mapping of $(\mathfrak{M}, \mathcal{M})$ or $(\mathfrak{N}, \mathcal{N})$ into itself for every $B \in \mathcal{S}$.*

Proof. The first assertion is true for simple f by the definition of $\mathcal{M}$ and $\mathcal{N}$, and it follows in general by monotone convergence. The second assertion follows from the first one by another application of the definition of $\mathcal{M}$. □

Lemma 1.4. *Let $\mathcal{I} \in \mathcal{B}$ be a semiring with $\hat{\sigma}(\mathcal{I}) = \mathcal{B}$. Then $\mathcal{M}$ and $\mathcal{N}$ are generated by the mappings $\mu \to \mu I$, $I \in \mathcal{I}$, and also by $\mu \to \mu f$, $f \in \mathcal{F}_c$.*

Proof. It is enough to consider $\mathcal{M}$, the arguments for $\mathcal{N}$ being similar. Let $\mathcal{M}'$ be the σ-algebra in $\mathfrak{M}$ generated by the mappings $\mu \to \mu I$, $I \in \mathcal{I}$, and note that $\mathcal{I} \subset \mathcal{B}$ implies $\mathcal{M}' \subset \mathcal{M}$. Thus it remains to prove that $\mathcal{M} \subset \mathcal{M}'$, i.e. that the mapping $\mu \to \mu B$ is $\mathcal{M}'$-measurable for every $B \in \mathcal{B}$. For this purpose, define

$$\mathcal{D} = \{B \in \mathcal{B}: \mu \to \mu B \text{ is } \mathcal{M}'\text{-measurable}\},$$

and note that $\mathcal{D}$ is closed under bounded monotone limits. Furthermore, $\mathcal{D}$ contains the ring $\mathcal{C}$ of all finite unions of $\mathcal{I}$-sets, since every such union may be taken to be disjoint. Hence we may conclude from 15.2.2 that $\mathcal{D} \supset \hat{\sigma}(\mathcal{C}) = \hat{\sigma}(\mathcal{I}) = \mathcal{B}$, which yields the desired measurability of $\mu \to \mu B$, $B \in \mathcal{B}$.

We next consider the σ-algebra $\mathcal{M}''$ generated by all mappings $\mu \to \mu f$, $f \in \mathcal{F}_c$, and note that $\mathcal{M}'' \subset \mathcal{M}$ by Lemma 1.3. To prove the converse relation, let $C \subset \mathfrak{S}$ be compact, and choose a sequence $f_1, f_2, \ldots \in \mathcal{F}_c$ satisfying $f_n \downarrow 1_C$, (cf. 15.6.1). Then

$\mu f_n \downarrow \mu C$, $\mu \in \mathfrak{M}$, by dominated convergence, which proves the $\mathcal{M}''$-measurability of $\mu \to \mu C$. We may now complete the proof as in case of Lemma 1.2, defining for fixed compact C

$$\mathcal{D} = \{B \in \mathcal{S}: \mu \to \mu(B \cap C) \text{ is } \mathcal{M}''\text{-measurable}\}.$$ □

Lemma 1.5. $\mathcal{N} \subset \mathcal{M}$.

Proof. By the definitions of $\mathcal{M}$ and $\mathcal{N}$ we have

$$\mathcal{N} \subset \mathcal{M} \cap \mathfrak{N} = \{M \cap \mathfrak{N}: M \in \mathcal{M}\},$$

so it is enough to prove that $\mathfrak{N} \in \mathcal{M}$. Let $\mathcal{U} \subset \mathcal{B}$ be an arbitrary countable DC-ring, and define

$$M = \{\mu \in \mathfrak{M}: \mu U \in \mathbb{Z}_+,\ U \in \mathcal{U}\}.$$

We intend to show that $\mathfrak{N} = M$; since clearly $M \in \mathcal{M}$, this will complete the proof. Now $\mathfrak{N} \subset M$ holds trivially, so it remains to prove that $M \subset \mathfrak{N}$. For this purpose, define for fixed $\mu \in M$

$$\mathcal{D} = \{B \in \mathcal{B}: \mu B \in \mathbb{Z}_+\}.$$

Since $\mathcal{D}$ is closed under bounded monotone limits and contains $\mathcal{U}$, it is seen from 15.2.2 and Lemma 1.2 that $\mathcal{D} \supset \hat{\sigma}(\mathcal{U}) = \mathcal{B}$. Thus $\mu \in \mathfrak{N}$, so we have indeed $M \subset \mathfrak{N}$.

1.2. Random measures and point processes

By a *random measure* or a *point process* on $\mathfrak{S}$ we mean any measurable mapping of some fixed probability space $(\Omega, \mathcal{A}, \mathsf{P})$ into $(\mathfrak{M}, \mathcal{M})$ or $(\mathfrak{N}, \mathcal{N})$ respectively. By Lemma 1.5, a point process may alternatively be considered as an $\mathfrak{N}$-valued random measure, and conversely any a.s. $\mathfrak{N}$-valued random measure coincides a.s. with a point process. Thus we shall make no difference in the sequel between point processes and a.s. $\mathfrak{N}$-valued random measures. Similarly, we shall allow a random measure to take values outside $\mathfrak{M}$ on a set $A \in \mathcal{A}$ with $\mathsf{P}A = 0$.

Lemma 1.6. *The class of random measures (or point processes) on $\mathfrak{S}$ is closed under addition and under multiplication by $\mathbb{R}_+$-valued (or $\mathbb{Z}_+$-valued respectively) random variables. Furthermore, a series $\sum_j \xi_j$ of random measures (or point processes) is itself a random measure (or point process) iff $\sum_j \xi_j B < \infty$ a.s. for all $B \in \mathcal{B}$.*

Proof. The first assertion follows immediately from the definition of $\mathcal{M}$ ($\mathcal{N}$) and the fact that the class of random variables is closed under addition and multiplication. As for the second assertion, it is seen by monotone convergence that $\sum_j \xi_j$ is σ-additive on $\mathcal{B}$, and hence measure-valued. Moreover, the necessity of our condition follows from the fact that $\xi B < \infty$ a.s. for any random measure ξ and any $B \in \mathcal{B}$. Suppose conversely that $\sum_j \xi_j B < \infty$ a.s., $B \in \mathcal{B}$. Considering this inequality for all sets B belonging to some countable covering class, it is seen that the exceptional P-null set may be taken to be independent of B. Thus $\sum_j \xi_j \in \mathfrak{M}$ ($\mathfrak{N}$) a.s. Finally, the measurability of $\sum_j \xi_j$ follows from the fact that $\sum_j \xi_j B$ is a random variable for every $B \in \mathcal{B}$. □

The *distribution* of a random measure or point process ξ is by definition the probability measure $\mathsf{P}\xi^{-1}$ on $(\mathfrak{M}, \mathcal{M})$ or $(\mathfrak{N}, \mathcal{N})$ given by

$$(\mathsf{P}\xi^{-1})M = \mathsf{P}(\xi^{-1}M) = \mathsf{P}\{\xi \in M\}\,, \qquad M \in \mathcal{M} \text{ or } \mathcal{N}\,.$$

We further define the *intensity* $\mathsf{E}\xi$ of ξ as the set function

$$(\mathsf{E}\xi)B = \mathsf{E}(\xi B)\,, \qquad B \in \mathcal{B}\,,$$

where E denotes the expectation or integral w.r.t. P. Note that $\mathsf{E}\xi$ is always a measure, though it need not belong to $\mathfrak{M}$. We finally define the *L-transform* (L for Laplace) L_ξ of ξ by

$$L_\xi(f) = \mathsf{E}\,e^{-\xi f}\,, \qquad f \in \mathcal{F}\,.$$

Note in particular that, for any $k \in \mathbb{N}$ and $B_1, \dots, B_k \in \mathcal{B}$, the function

$$L_\xi(\sum_j t_j 1_{B_j}) = \mathsf{E}\exp(-\sum_j t_j \xi B_j) = L_{\xi B_1, \dots, \xi B_k}(t_1, \dots, t_k)\,, \qquad t_1, \dots, t_k \in \mathbb{R}_+\,,$$

is the elementary L-transform of the random vector $(\xi B_1, \dots, \xi B_k)$.

Many important random measure distributions are defined most easily by means of mixing. In the present context, the general measurability requirement 15.3.2 can be relaxed as follows.

Lemma 1.7. *Let $(\mathfrak{T}, \mathcal{T}, \mathsf{Q})$ be a probability space and let ξ_ϑ, $\vartheta \in \mathfrak{T}$, be a family of random measures on $\mathfrak{S}$. Then the mixture of $\mathsf{P}\xi_\vartheta^{-1}$ w.r.t. Q exists iff $L_{\xi_\vartheta}(f)$ is $\mathcal{T}$-measurable in ϑ for every $f \in \mathcal{F}_c$.*

Proof. By 15.3.2. the mixture exists iff $\mathsf{P}\{\xi_\vartheta \in M\}$ is $\mathcal{T}$-measurable for every $M \in \mathcal{M}$, so let us first assume that this condition is fulfilled. Then $\mathsf{P}\{\xi_\vartheta f \leqq x\}$ is $\mathcal{T}$-measurable for every $f \in \mathcal{F}_c$ and $x \in \mathbb{R}_+$ according to Lemma 1.3, and it follows by the definition of the integral that $L_{\xi_\vartheta}(f) = \mathsf{E}e^{-\xi_\vartheta f}$ is measurable for every $f \in \mathcal{F}_c$.

Conversely, suppose that $L_{\xi_\vartheta}(f)$ is $\mathcal{T}$-measurable for every $f \in \mathcal{F}_c$. Then so is

$$L_{\xi_\vartheta f_1, \dots, \xi_\vartheta f_k}(t_1, \dots, t_k) = L_{\xi_\vartheta}(\sum_j t_j f_j)$$

for every $k \in \mathbb{N}$, $f_1, \dots, f_k \in \mathcal{F}_c$ and $t_1, \dots, t_k \in \mathbb{R}_+$, and scrutinizing the proof of the uniqueness theorem 15.5.1 for multidimensional L-transforms, it is seen that $\mathsf{P}\{\xi_\vartheta f_1 \leqq x_1, \dots, \xi_\vartheta f_k \leqq x_k\}$ is $\mathcal{T}$-measurable for any $k \in \mathbb{N}$, $f_1, \dots, f_k \in \mathcal{F}_c$ and $x_1, \dots, x_k \in \mathbb{R}_+$. We now introduce the class $\mathcal{D}$ of all sets $M \in \mathcal{M}$ such that $\mathsf{P}\{\xi_\vartheta \in M\}$ is $\mathcal{T}$-measurable, and note that $\mathcal{D}$ contains $\mathfrak{M}$ and is closed under proper differences and monotone limits. Furthermore, it was seen above that $\mathcal{D}$ contains the class $\mathcal{C}$ of all sets of the form

$$\{\mu \in \mathcal{M}: \mu f_1 \leqq x_1, \dots, \mu f_k \leqq x_k\}\,, \quad k \in \mathbb{N}\,, \quad f_1, \dots, f_k \in \mathcal{F}_c\,, \quad x_1, \dots, x_k \in \mathbb{R}_+\,,$$

and the latter class being closed under finite intersections, it follows by 15.2.1 that $\mathcal{D} \supset \sigma(\mathcal{C})$. Since $\sigma(\mathcal{C}) = \mathcal{M}$ by Lemma 1.4, this means that $\mathsf{P}\{\xi_\vartheta \in M\}$ is indeed $\mathcal{T}$-measurable for every $M \in \mathcal{M}$. □

1.3. Basic processes and mappings

For any fixed $s \in \mathfrak{S}$, we define the *Dirac measure* $\delta_s \in \mathfrak{N}$ by $\delta_s B = 1_B(s)$, $B \in \mathcal{B}$. The mapping $s \to \delta_s$ is clearly measurable $(\mathfrak{S}, \mathcal{S}) \to (\mathfrak{N}, \mathcal{N})$, and in particular δ_τ is a point process on $\mathfrak{S}$ for any random element τ in $(\mathfrak{S}, \mathcal{S})$. Writing $\omega = \mathsf{P}\tau^{-1}$, it is seen that δ_τ has intensity $\mathsf{E}\delta_\tau = \mathsf{P}\tau^{-1} = \omega$ and L-transform

$$\mathsf{E}\,e^{-\delta_\tau f} = \mathsf{E}\,e^{-f(\tau)} = \omega\,e^{-f}\,, \qquad f \in \mathcal{F}\,. \tag{1.1}$$

Next suppose that $n \in \mathbb{Z}_+$, and let $\tau_1, \dots, \tau_n$ be independent random elements in $\mathfrak{S}$ with common distribution ω. We shall say that a point process ξ on $\mathfrak{S}$ is a *sample process* with intensity $n\omega$, if ξ has the same distribution as $\delta_{\tau_1} + \dots + \delta_{\tau_n}$. By (1.1) and the assumed independence, ξ has then the L-transform

$$\mathsf{E}\exp(-\sum_{j=1}^n \delta_{\tau_j} f) = \prod_{j=1}^n \mathsf{E}\exp(-\delta_{\tau_j} f) = (\omega e^{-f})^n\,, \qquad f \in \mathcal{F}\,. \tag{1.2}$$

By Lemma 1.7, we may consider $n = \nu$ as a $\mathbb{Z}_+$-valued random variable and mix w.r.t. its distribution to obtain a *mixed sample process* with intensity $(\mathsf{E}\nu)\omega$ and L-transform

$$\mathsf{E}\,e^{-\xi f} = \mathsf{E}(\omega\,e^{-f})^\nu = \psi(\omega\,e^{-f})\,, \qquad f \in \mathcal{F}\,. \tag{1.3}$$

Here ψ denotes the *(probability) generating function* of ν, i.e.

$$\psi(s) = \mathsf{E}s^\nu\,, \qquad s \in [0, 1]\,.$$

In the particular case when ν is Poisson with mean $a \geqq 0$, ψ is given by

$$\psi(s) = e^{-a} \sum_{n=0}^\infty \frac{a^n s^n}{n!} = e^{-a(1-s)}\,, \qquad s \in [0, 1]\,,$$

and (1.3) becomes, with $\lambda = a\omega$,

$$\mathsf{E}\,e^{-\xi f} = e^{-a(1-\omega\,e^{-f})} = e^{-\lambda(1-e^{-f})}\,, \qquad f \in \mathcal{F}\,, \tag{1.4}$$

where we have used the fact that $\omega\mathfrak{S} = 1$. A point process with this distribution is called a *Poisson process* with intensity λ. In this case λ is bounded, but we may also construct Poisson processes with unbounded intensity $\lambda \in \mathfrak{M}$. For this purpose, let $S_1, S_2, \dots \in \mathcal{B}$ be any disjoint partition of $\mathfrak{S}$ into bounded sets, (cf. 15.6.1 for existence). Since the corresponding restrictions $S_j\lambda$, $j \in \mathbb{N}$, of λ are bounded, there exist some independent Poisson processes $\xi_1, \xi_2, \dots$ on $\mathfrak{S}$ with these measures as intensities. Moreover,

$$\sum_j \mathsf{E}\xi_j B = \sum_j (S_j\lambda) B = \lambda B < \infty\,, \qquad B \in \mathcal{B}\,,$$

so the series $\sum \xi_j$ converges by Lemma 1.6 to some point process ξ on $\mathfrak{S}$. Applying (1.4) to each ξ_j and using the assumed independence, we obtain for any $f \in \mathcal{F}$

$$\mathsf{E}\,e^{-\xi f} = \prod_j \mathsf{E}\,e^{-\xi_j f} = \prod_j \exp\{-(S_j\lambda)(1-e^{-f})\}$$
$$= \exp\{-\sum_j (S_j\lambda)(1-e^{-f})\} = e^{-\lambda(1-e^{-f})}\,,$$

so (1.4) remains true. (The formal calculations here and in similar places are justified sy the fact that all quantities involved are non-negative.) Any point process ξ with b-transform $e^{-\lambda(1-e^{-f})}$ will henceforth be called a *Poisson process with intensity* λ. Ls will be seen from Theorem 3.1, its distribution $\mathsf{P}\xi^{-1}$ is uniquely determined by λ.

If ξ is a Poisson process on $\mathfrak{S}$ with intensity λ and if $B_1, \dots, B_k \in \mathcal{B}$ are disjoint, we get for any $t_1, \dots, t_k \in \mathbb{R}_+$

$$L_{\xi B_1, \dots, \xi B_k}(t_1, \dots, t_k) = L_\xi(\sum_j t_j 1_{B_j}) = \exp\{-\lambda[1 - \exp(-\sum_j t_j 1_{B_j})]\}$$
$$= \exp\{-\lambda \sum_j 1_{B_j}(1 - e^{-t_j})\} = \prod_j \exp\{-\lambda 1_{B_j}(1 - e^{-t_j})\}$$
$$= \prod_j L_\xi(t_j 1_{B_j}) = \prod_j L_{\xi B_j}(t_j)\,.$$

By 15.5.1, this shows that ξ has *independent increments*, in the sense that $\xi B_1, \dots, \xi B_k$ are independent for any $k \in \mathbb{N}$ and disjoint $B_1, \dots, B_k \in \mathcal{B}$. This fact was actually the basis for the above construction of Poisson processes with unbounded intensity.

Let us next consider a Poisson process ξ_α with intensity $\alpha\lambda$, where $\alpha \in \mathbb{R}_+$ while $\lambda \in \mathfrak{M}$. By (1.4) and Lemma 1.7, we may consider α as an $\mathbb{R}_+$-valued random variable and mix w.r.t. its distribution, thus obtaining a *mixed Poisson process*, possessing the L-transform

$$\mathsf{E}\, e^{-\alpha\lambda(1-e^{-f})} = L_\alpha(\lambda(1-e^{-f}))\,, \quad f \in \mathcal{F}\,. \tag{1.5}$$

More generally, it is seen from (1.4) and Lemmas 1.3 and 1.7 that the intensity $\lambda = \eta$ of a Poisson process may be considered as a random measure on $\mathfrak{S}$. In this way we obtain by mixing a *Cox process* ξ *directed by* η, possessing the L-transform

$$L_\xi(f) = \mathsf{E}\, e^{-\eta(1-e^{-f})} = L_\eta(1-e^{-f})\,, \quad f \in \mathcal{F}\,. \tag{1.6}$$

Table 1. Some basic point processes and random measures.

process ξ	based on	$\mathsf{E}\xi$	$\mathsf{P}\{\xi B = 0\}$	$L_\xi(f)$
sample	ω, n	$n\omega/\omega\mathfrak{S}$	$(\omega B^c/\omega\mathfrak{S})^n$	$(\omega\, e^{-f}/\omega\mathfrak{S})^n$
mixed sample	$\omega, \nu(\psi)$	$(\mathsf{E}\nu)\,\omega/\omega\mathfrak{S}$	$\psi(\omega B^c/\omega\mathfrak{S})$	$\psi(\omega\, e^{-f}/\omega\mathfrak{S})$
Poisson	λ	λ	$e^{-\lambda B}$	$e^{-\lambda(1-e^{-f})}$
mixed Poisson	λ, α	$(\mathsf{E}\alpha)\,\lambda$	$L_\alpha(\lambda B)$	$L_\alpha(\lambda(1-e^{-f}))$
Cox	η	$\mathsf{E}\eta$	$\mathsf{E}e^{-\eta B}$	$L_\eta(1-e^{-f})$
compound	η, β	$(\mathsf{E}\beta)\,\mathsf{E}\eta$	$\mathsf{E}(\mathsf{P}\{\beta = 0\})^{\eta B}$	$L_\eta(-\log L_\beta \circ f)$
thinning	η, p	$p\mathsf{E}\eta$	$\mathsf{E}(1-p)^{\eta B}$	$L_\eta(-\log[1-p(1-e^{-f})])$
randomization	η, λ	$\mathsf{E}\eta \times \lambda$	$L_\eta(-\log \lambda B^c)$	$L_\eta(-\log \lambda e^{-f})$

Let us now introduce compound point processes as follows. As will be seen in Lemma 2.1 below, there exists for every fixed $\mu \in \mathfrak{N}$ some finite or infinite sequence $t_1, t_2, \ldots \in \mathfrak{S}$ such that $\mu = \sum \delta_{t_j}$. Assuming $\beta, \beta_1, \beta_2, \ldots$ to be independent and identically distributed $\mathbb{R}_+$-valued random variables, it follows by Lemma 1.6 that $\xi = \sum \beta_j \delta_{t_j}$ is a random measure on $\mathfrak{S}$, and by the assumed independence, its L-transform becomes for $f \in \mathcal{F}$ (writing $\circ$ for composition)

$$\begin{aligned}\mathsf{E}\exp\Big(-\sum_j \beta_j f(t_j)\Big) &= \prod_j \mathsf{E}\exp(-\beta f(t_j)) = \prod_j L_\beta \circ f(t_j) \\ &= \exp \sum_j \log L_\beta \circ f(t_j) = \exp(\mu \log L_\beta \circ f)\,.\end{aligned}$$

According to Lemmas 1.3 and 1.7, we may mix here w.r.t. $\mu = \eta$ regarded as a point process on $\mathfrak{S}$, and in this way we obtain a β*-compound of* η, possessing the L-transform

$$\mathsf{E}\exp(\eta \log L_\beta \circ f) = L_\eta(-\log L_\beta \circ f)\,, \quad f \in \mathcal{F}\,. \tag{1.7}$$

In the particular case when β equals either 0 or 1, these values being attained with probabilities and $1-p$ and p respectively, we get a p*-thinning of* η, and the L-transform in (1.7) reduces to

$$L_\eta(-\log[p\, e^{-f} + (1-p)]) = L_\eta(-\log[1-p(1-e^{-f})])\,, \quad f \in \mathcal{F}\,. \tag{1.8}$$

Intuitively, a p-thinning of η is obtained by deleting the unit atoms of η independently with probability $1-p$ each. (Here and in similar cases, the phrase "unit atoms" is

intended to mean that an atom of size $n > 1$ should be regarded as the sum of n atoms of size 1.)

Starting from a fixed measure $\mu = \sum \delta_{t_j}$ as above, we next consider the point process $\xi = \sum \delta_{(t_j, \sigma_j)}$ on $\mathfrak{S} \times \mathbb{R}$, where $\sigma_1, \sigma_2, \ldots$ are independent random variables with common distribution λ. Letting $f \in \mathcal{F}(\mathfrak{S} \times \mathbb{R})$ be arbitrary, we obtain

$$\begin{aligned}\mathsf{E}\exp\Big(-\sum_j f(t_j, \sigma_j)\Big) &= \prod_j \mathsf{E}\exp(-f(t_j, \sigma_j)) = \prod_j \lambda \exp(-f(t_j, \cdot)) \\ &= \exp \sum_j \log \lambda \exp(-f(t_j, \cdot)) = \exp \mu \log \lambda\, e^{-f}\,.\end{aligned}$$

In this case, mixing w.r.t. $\mu = \eta$ yields a λ*-randomization* ξ of η with L-transform

$$L_\xi(f) = L_\eta(-\log \lambda\, e^{-f})\,. \tag{1.9}$$

In particular, ξ is said to be a *uniform* randomization of η, when λ equals Lebesgue measure on $[0, 1]$.

We finally point out that the above method of constructing the basic point processes does not depend on the special assumptions on $\mathfrak{S}$ made above. In particular, Poisson processes with arbitrary σ-finite intensities may be constructed in any measurable space. This possibility will be useful in Chapter 6.

1.4. Exercises

1.1. Let $\mathcal{J} \subset \mathcal{B}$ be a semiring satisfying $\hat{\sigma}(\mathcal{J}) = \mathcal{B}$. Show that Lemma 1.7 remains true with $\mathcal{F}_c$ replaced by the class of all *simple functions over* $\mathcal{J}$, i.e. of all functions of the form $\sum_{j=1}^k t_j 1_{I_j}$ with arbitrary $k \in \mathbb{N}$, $t_1, \ldots, t_k \in \mathbb{R}_+$ and $I_1, \ldots, I_k \in \mathcal{J}$.

1.2. Show that, if ξ is a Cox process directed by η, then ξ and η have simultaneously independent increments. Prove the corresponding fact for β-compounds with $\beta \overset{d}{\neq} 0$. (*Hint*: Use 15.5.1.)

1.3. Let ξ be a random measure on $\mathfrak{S}$ and let $\mathcal{J} \subset \mathcal{B}$ be a semiring satisfying $\hat{\sigma}(\mathcal{J}) = \mathcal{B}$. Show that ξ has independent increments iff $\xi I_1, \ldots, \xi I_k$ are independent for any $k \in \mathbb{N}$ and disjoint $I_1, \ldots, I_k \in \mathcal{J}$. (Cf. Matthes et al. (1978), p. 16. *Hint*: Extend the independence property, first to the ring generated by $\mathcal{J}$, and then by means of 15.2.2 to $\mathcal{B}$, considering one component at a time.)

1.4. Verify the expressions for $\mathsf{E}\xi$ and $\mathsf{P}\{\xi B = 0\}$ given in Table 1. (*Hint*: Make use of mixing or apply the formulae

$$\mathsf{E}\xi B = -\frac{d}{dt} L_{\xi B}(t)\Big|_{t=0}\,, \qquad \mathsf{P}\{\xi B = 0\} = \lim_{t\to\infty} L_{\xi B}(t)\,, \qquad B = \mathcal{B}\,.)$$

1.5. Let ξ be a random measure on $\mathfrak{S}$ and let $\mathcal{U} \subset \mathcal{B}$ be a ring satisfying $\hat{\sigma}(\mathcal{U}) = \mathcal{B}$. Show that there exists for every $B \in \mathcal{B}$ some sequence $U_1, U_2, \ldots \in \mathcal{U}$ such that $\xi U_n \xrightarrow{P} \xi B$. (*Hint*: Use 15.2.2. Cf. Matthes et al. (1978), p. 29.) Extend this result to several dimensions: For every $k \in \mathbb{N}$ and disjoint $B_1, \ldots, B_k \in \mathcal{B}$ there exists some sequence $(U_{n1}, \ldots, U_{nk}) \in \mathcal{U}^k$, $n \in \mathbb{N}$, such that $U_{n1}, \ldots, U_{nk}$ are disjoint for each $n \in \mathbb{N}$ and moreover $(\xi U_{n1}, \ldots, \xi U_{nk}) \xrightarrow{P} (\xi B_1, \ldots, \xi B_k)$.

1.6. For fixed $p \in (0, 1)$, let ξ be a p-thinning of some point process η on $\mathfrak{S}$. Consider a fixed $B \in \mathcal{B}$. Prove that $\mathsf{E}(p^{-1}-1)^{\xi B} < \infty$ iff $\mathsf{E}(2(1-p))^{\eta B} < \infty$, and that in this case $\mathsf{P}\{\eta B = 0\} = \mathsf{E}(1-p^{-1})^{\xi B}$. Note that the two equivalent conditions are automatically fulfilled when $p \geqq 1/2$. (*Hint*: Consider the generating function of ξB as a Taylor series around $1-p$ and determine its radius of convergence.)

1.7. Consider the space of all σ-finite measures on $\mathfrak{S}$, and form a space $\mathfrak{M}'$ by identifying measures μ_1 and μ_2 with $\mu_1 f = \mu_2 f$, $f \in \mathcal{F}_c$. Let $\mathcal{M}'$ be the σ-algebra in $\mathfrak{M}'$ generated by the mappings $\mu \to \mu f$, $f \in \mathcal{F}_c$. Show that $\mathfrak{M} \subset \mathfrak{M}'$ and $\mathcal{M} \subset \mathcal{M}'$. Prove Lemma 1.7 for random elements in $(\mathfrak{M}', \mathcal{M}')$.

1.8. Let $\mathcal{J} \subset \mathcal{B}$ be a DC-semiring and write $\mathfrak{M}_{\mathcal{J}}$ for the space of σ-finite measures on $\mathfrak{S}$ where μ_1 and μ_2 are identified whenever $\mu_1 I = \mu_2 I$, $I \in \mathcal{J}$. Let $\mathcal{M}_{\mathcal{J}}$ be the σ-algebra generated by all mappings $\mu \to \mu I$, $I \in \mathcal{J}$. Show that $\mathfrak{M} \subset \mathfrak{M}_{\mathcal{J}}$ and $\mathcal{M} \subset \mathcal{M}_{\mathcal{J}}$, and further that the mapping $\mu \to \mu f$ is unique and measurable for every $f \in \mathcal{F}_c$. Prove a version of Lemma 1.7 for random elements in $(\mathfrak{M}_{\mathcal{J}}, \mathcal{M}_{\mathcal{J}})$.

1.9. Let the space $\mathfrak{S}$ and $\mathfrak{S}'$ be lcscH, and let $f: \mathfrak{S} \to \mathfrak{S}'$ be such that $f^{-1}(\mathcal{B}(\mathfrak{S}')) \subset \mathcal{B}(\mathfrak{S})$. Show that $\mu \to \mu f^{-1}$ then defines a measurable mapping from $\mathfrak{M}(\mathfrak{S})$ to $\mathfrak{M}(\mathfrak{S}')$. Thus ξf^{-1} is a random measure (point process) on $\mathfrak{S}'$ whenever the corresponding thing is true for ξ on $\mathfrak{S}$. Prove also that

$$L_{\xi f^{-1}}(g) = L_\xi(g \circ f)\,, \qquad g \in \mathcal{F}(\mathfrak{S}')\,.$$

1.10. Say that $\mathcal{C} \subset \mathcal{B}$ is a *separating class* if, for every pair $F, G \in \mathcal{B}$ with $F \subset G$ and such that F is closed while G is open, there exists some $C \in \mathcal{C}$ with $F \subset C \subset G$. Show that every DC-ring is separating.

1.11. Show how (1.8) may be deduced from (1.9).

2. Sample realizations

2.1. Decompositions

Let $\mathfrak{M}_d$ denote the class of all *diffuse* (or *non-atomic*) measures in $\mathfrak{M}$, and define $R'_+ = (0, \infty)$.

Lemma 2.1. *Every measure $\mu \in \mathfrak{M}$ may be written in the form*

$$\mu = \mu_d + \sum_{j=1}^{k} b_j \delta_{t_j} \tag{2.1}$$

for some $\mu_d \in \mathfrak{M}_d$, $k \in Z_+ \cup \{\infty\}$, $b_1, b_2, \ldots \in R'_+$ and $t_1, t_2, \ldots \in \mathfrak{S}$, and this decomposition is unique apart from the order of terms, provided the t_j are assumed to be distinct. In this case, $\mu \in \mathfrak{N}$ iff $\mu_d = 0$ and $b_1, b_2, \ldots \in N$.

Proof. Since μ is σ-finite, it can have at most countably many atoms of size greater than some fixed $a > 0$, and hence the total number of atoms is at most countable. Choose an arbitrary enumeration $b_1\delta_{t_1}, b_2\delta_{t_2}, \ldots$, and verify that $\mu_d = \mu - \sum_j b_j \delta_{t_j}$ is a non-atomic measure. The uniqueness assertion follows by an obvious identification procedure.

Now suppose that $\mu \in \mathfrak{N}$. Since the one-point sets of $\mathfrak{S}$ clearly belong to $\mathcal{B}$, it follows immediately that $b_1, b_2, \ldots \in N$, and hence that $\mu_d \in \mathfrak{N}$. To see that this implies $\mu_d = 0$, note that every point $s \in \mathfrak{S}$ has a neighbourhood $G_s \in \mathcal{B}$ satisfying $\mu_d G_s = \mu_d\{s\} = 0$. Now every compact set may be covered by finitely many sets G_s, and we get $\mu_d B = 0$, $B \in \mathcal{B}$, as desired. □

Given any $\mu \in \mathfrak{M}$ and $a \in (0, \infty]$, we define the measures $\mu_a^* \in \mathfrak{N}$ and $\mu_a' \in \mathfrak{M}$ by

$$\mu_a^* B = \sum_{s \in B} 1_{[a,\infty)}(\mu\{s\})\,, \quad B \in \mathcal{B}\,,$$

$$\mu_a' B = \mu B - \sum_{s \in B} \mu\{s\}\, 1_{[a,\infty)}(\mu\{s\})\,, \quad B \in \mathcal{B}\,.$$

Note that μ_a^* is obtained by counting each μ-atom of size $\geqq a$ once, while μ_a' is obtained from μ by subtracting all such atoms. When $\mu \in \mathfrak{N}$, we shall write $\mu_1^* = \mu^*$ and say that μ is *simple* if $\mu_2^* = 0$, i.e. if all atoms of μ have unit size.

We further introduce the notion of *null-array of partitions* of some fixed set $B \in \mathcal{B}$. By this we mean an array $\{B_{nj}\}$ of $\mathcal{B}$-sets, such that for fixed $n \in N$ the B_{nj} form a finite disjoint partition of B and such that $\max_j |B_{nj}| \to 0$, where $|\cdot|$ denotes the diameter in any fixed metric ϱ. (Arguing as in Lemma 1.1, it is seen that the last condition is independent of the choice of ϱ.) Let us further say that the partitions $\{B_{nj}\} \subset \mathcal{B}$ of $\mathfrak{S}$ form a null-array, if for every fixed $C \in \mathcal{B}$ and $n \in N$, only finitely many sets B_{nj} intersect C, and their diameters tend uniformly to zero as $n \to \infty$.

The next lemma is basic.

Lemma 2.2. *Let $\{B_{nj}\} \subset \mathscr{B}$ be a null-array of partitions of some fixed set $B \in \mathscr{B}$. Then*

$$\lim_{n\to\infty} \sum_j 1_{[a,\infty)}(\mu B_{nj}) = \mu_a^* B\,, \qquad \mu \in \mathfrak{M}\,, \qquad a \in R'_+\,. \tag{2.2}$$

Proof. Since $\max_j |B_{nj}| \to 0$, all μ-atoms in B of size $\geqq a$ will ultimately lie in different partitioning sets, and so we get for large $n \in N$

$$\sum_j 1_{[a,\infty)}(\mu B_{nj}) \geqq \mu_a^* B\,. \tag{2.3}$$

Now suppose that the inequality in (2.3) is strict for infinitely many $n \in N$, say for $n \in N' \subset N$. Then there exist some indices j_n, $n \in N'$, such that

$$\mu'_a B_{n,j_n} \geqq a\,, \qquad n \in N'. \tag{2.4}$$

Choosing arbitrary $s_n \in B_{n,j_n}$, $n \in N'$, it is seen from the compactness of B^- that there exists some subsequence $N'' \subset N'$ such that $s_n \to$ some $s \in B^-$ $(n \in N'')$, and since $|B_{n,j_n}| \to 0$, it follows from (2.4) that $\mu'_a G \geqq a$ for every open set $G \subset B$ containing s. But then $\mu'_a\{s\} \geqq a$, which contradicts the definition of μ'_a. This shows that the inequality in (2.3) can be strict for at most finitely many $n \in N$, and hence completes the proof. □

We may now extend the decomposition (2.1) to random measures.

Lemma 2.3. *The decomposition* (2.1) *may be chosen such that $\mu_d, k, b_1, b_2, \ldots$ and $t_1, t_2, \ldots$ are measurable functions of μ.*

Proof. Let $\mathfrak{N}' = \mathfrak{N}'(\mathfrak{S})$ be the set of all measures $\mu \in \mathfrak{N}$ with $\mu_2^* = 0$, and conclude from Lemma 2.2 that $\mathfrak{N}' \in \mathscr{N}$. Our first aim is to prove the assertion for $\mathfrak{N}'$. For this purpose, choose a metric ϱ and a dense sequence $s_1, s_2, \ldots$ in $\mathfrak{S}$, and note that $\varrho(s, s_n) = \varrho(t, s_n)$ for all $n \in N$ iff $s = t$. (In fact, assuming this condition to be true, we have either $s = s_n$ for some n, and then $\varrho(t, s_n) = \varrho(s, s_n) = 0$, which means that $t = s_n = s$; or else there exists some sequence $N' \subset N$ such that $s_n \to s$ $(n \in N')$, and then $\varrho(t, s_n) = \varrho(s, s_n) \to 0$, proving that $s_n \to t$, and again we may conclude that $s = t$.) Because of this fact, we may define a linear order in $\mathfrak{S}$ by writing $s < t$ whenever, for some $k \in N$,

$$\varrho(s, s_j) = \varrho(t, s_j)\,, \qquad j = 1, \ldots, k-1; \qquad \varrho(s, s_k) < \varrho(t, s_k)\,.$$

Let us further introduce a disjoint partition $B_1, B_2, \ldots \in \mathscr{B}$ of $\mathfrak{S}$. Since $\mu B_n < \infty$ for each $n \in N$, we may order the atoms $t_1, t_2, \ldots$ of μ, first according to their occurrance in $B_1, B_2, \ldots$, and then, within each B_n, w.r.t. their linear order. To show that the t_j, when ordered in this way, are measurable functions of μ, it is seen by induction that it suffices to consider t_1, i.e. to prove that $\{\mu\colon t_1 \in B\}$ belongs to $\mathscr{N}$ for each $B \in \mathscr{B}$, and this follows from the fact that

$$\{\mu\colon t_1 \in B\} = \bigcup_{n,k\in N} \Big[\Big(\bigcap_{j=1}^{k-1} \bigcap_{r\in Q_+} \{\mu\colon \mu(B_n \cap S_j(r)) \neq 1\}\Big) \cap \Big(\bigcup_{r\in Q_+} \{\mu\colon \mu(B \cap B_n \cap S_k(r)) = \mu(B_n \cap S_k(r)) = 1\}\Big)\Big]\,,$$

where $S_j(r) = \{s \in \mathfrak{S}\colon \varrho(s, s_j) < r\}$ while Q_+ denotes the set of rational numbers in R_+.

Let us now consider the case of general $\mu \in \mathfrak{M}$, and define the set function $\xi = \xi(\mu)$ by

$$\xi(B \times [a, \infty)) = \mu_a^* B\,, \qquad B \in \mathscr{B}\,, \qquad a \in R'_+\,. \tag{2.5}$$

By the Caratheodory extension theorem, $\xi(\mu)$ extends to a measure in $\mathfrak{N}'(\mathfrak{S} \times R'_+)$, and it follows by the first part of the proof that ξ may be written in the form

$$\xi(\mu) = \sum_{j=1}^k \delta_{(t_j, b_j)}\,, \tag{2.6}$$

where k and the pairs (t_j, b_j) are measurable functions of ξ. But from Lemmas 1.3 and 2.2 it is seen that $\xi(\mu)$ is a measurable function of μ, and hence so are $k, b_1, b_2, \ldots$ and $t_1, t_2, \ldots$ Moreover, it is easily seen from (2.5) and (2.6) that (2.1) is valid with this choice of parameters. Finally, the measurability of μ_d follows from (2.1) by Lemmas 1.3 and 1.5. □

It should be noticed that the measurable decomposition in Lemma 2.3 is by no means unique, not even apart from the order of terms (cf. Exercise 2.7). However, when considering random measures ξ on $\mathfrak{S}$ with $\xi\mathfrak{S} < \infty$ a.s. (which holds automatically when $\mathfrak{S}$ is compact), it is sometimes natural to require the atom sizes $\beta_1, \beta_2, \ldots$ to be taken in order of decreasing magnitude and, whenever two or more β_j coincide, to order the corresponding atom positions τ_j at random. (We shall always assume our basic probability space to be rich enough to support any randomization we need.) Though this procedure does not lead to a unique decomposition

$$\xi = \xi_d + \sum_{j=1}^{\nu} \beta_j \delta_{\tau_j}\,, \tag{2.7}$$

it does lead to a unique joint distribution of the random elements $\xi_d, \nu, \beta_1, \beta_2, \ldots$ and $\tau_1, \tau_2, \ldots$ A decomposition (2.7) with the above properties will be needed in Chapter 9.

2.2. Intensities and regularity

Theorem 2.4. *Let ξ be a random measure on $\mathfrak{S}$, and let $a \in R'_+$ and $B \in \mathscr{B}$ be such that $\mathsf{E}\xi'_a B < \infty$. Further suppose that $\{B_{nj}\} \subset \mathscr{B}$ is a null-array of partitions of B. Then*

$$\lim_{n\to\infty} \sum_j \mathsf{P}\{a \leqq \xi B_{nj} < b\} = \mathsf{E}(\xi_a^* - \xi_b^*)\, B\,, \qquad b \in (a, \infty]\,. \tag{2.8}$$

Proof. Put for brevity $\xi_a^* - \xi_b^* = \xi_{[a,b)}^*$. Taking differences in Lemma 2.2, it is seen that

$$\lim_{n\to\infty} \sum_j 1_{[a,b)}(\xi B_{nj}) = \xi_{[a,b)}^* B\,, \tag{2.9}$$

and hence by Fatou's lemma

$$\mathsf{E}\xi_{[a,b)}^* B \leqq \liminf_{n\to\infty} \mathsf{E} \sum_j 1_{[a,b)}(\xi B_{nj}) = \liminf_{n\to\infty} \sum_j \mathsf{P}\{a \leqq \xi B_{nj} < b\}\,,$$

which proves (2.8) in the case when $\mathsf{E}\xi_{[a,b)}^* B = \infty$. Thus it remains to consider the case when $\mathsf{E}\xi'_a B$ and $\mathsf{E}\xi_{[a,b)}^* B$ are both finite. But then (2.8) follows from (2.9) by dominated convergence, since

$$\{a \leqq \xi B_{nj} < b\} \subset \{\xi_{[a,b)}^* B_{nj} \geqq 1\} \cup \{\xi'_a B_{nj} \geqq a\}\,,$$

and therefore

$$\begin{aligned}\sum_j 1_{[a,b)}(\xi B_{nj}) &\leqq \sum_j 1_{[1,\infty)}(\xi_{[a,b)}^* B_{nj}) \vee 1_{[a,\infty)}(\xi'_a B_{nj}) \\ &\leqq \sum_j 1_{[1,\infty)}(\xi_{[a,b)}^* B_{nj}) + \sum_j 1_{[a,\infty)}(\xi'_a B_{nj}) \\ &\leqq \sum_j \xi_{[a,b)}^* B_{nj} + \sum_j a^{-1}\xi'_a B_{nj} = \xi_{[a,b)}^* B + a^{-1}\xi'_a B\,. \end{aligned}$$

□

Given any covering class $\mathcal{I} \subset \mathcal{B}$ and any $a > 0$, we shall say that ξ is *a-regular* w.r.t. $\mathcal{I}$, if for every fixed $I \in \mathcal{I}$ there exists some array $\{I_{nj}\} \subset \mathcal{I}$ of finite covers of I (one for each $n \in \mathbb{N}$) such that

$$\lim_{n\to\infty} \sum_j \mathsf{P}\{\xi I_{nj} \geqq a\} = 0 . \tag{2.10}$$

Theorem 2.5. *Let ξ be a random measure on $\mathfrak{S}$, let $\mathcal{I} \subset \mathcal{B}$ be a covering class and let $a > 0$. Then $\xi_a^* = 0$ a.s. provided ξ is a-regular w.r.t. $\mathcal{I}$. The converse is also true if $\mathsf{E}\xi \in \mathfrak{M}$ and $\mathcal{I}$ is a DC-semiring.*

Proof. Let $I \in \mathcal{I}$ be fixed and let $\{I_{nj}\} \subset \mathcal{I}$ be an array of finite covers of I satisfying (2.10). Then

$$\mathsf{P}\{\xi_a^* I > 0\} \leqq \mathsf{P} \bigcup_j \{\xi I_{nj} \geqq a\} \leqq \sum_j \mathsf{P}\{\xi I_{nj} \geqq a\} \to 0 ,$$

which yields $\xi_a^* I = 0$ a.s. Thus the first assertion follows from the fact that $\mathcal{I}$ is covering. Conversely, suppose that $\xi_a^* = 0$ a.s. and $\mathsf{E}\xi \in \mathfrak{M}$, and let $\mathcal{I}$ be a DC-semiring. Choose for any fixed $I \in \mathcal{I}$ some null-array $\{I_{nj}\} \subset \mathcal{I}$ of partitions of I, and conclude from Theorem 2.4 that

$$\lim_{n\to\infty} \sum_j \mathsf{P}\{\xi I_{nj} \geqq a\} = \mathsf{E}\xi_a^* I = 0 .$$

Since I was arbitrary, this proves the asserted regularity of ξ. □

It is sometimes useful to replace the global regularity condition (2.10) by a local one. For a first step, note that if $\mathcal{I}$ is a DC-semiring and if $\lambda \in \mathfrak{M}$, then (2.10) holds if $\mathsf{P}\{\xi I \geqq a\} = o(\lambda I)$ as $|I| \to 0$, uniformly for all $I \in \mathcal{I}$ contained in an arbitrary compact set, or equivalently, if

$$\lim_{\varepsilon\to 0} \sup \left\{\frac{1}{\lambda I} \mathsf{P}\{\xi I \geqq a\} : I \in \mathcal{I} \cap I_0 , |I| < \varepsilon\right\} = 0 , \quad I_0 \in \mathcal{I} , \tag{2.11}$$

so in this case $\xi_a^* = 0$ a.s. (Here and below, $0/0$ is to be interpreted as 0.) We shall show that the same conclusion may be drawn even without the uniformity requirement. Let us say that a sequence of partitions is *nested* if it proceeds by successive refinements.

Theorem 2.6. *Let ξ be a random measure on $\mathfrak{S}$ and let $a > 0$. Then $\xi_a^* = 0$ a.s., provided there exist some DC-semiring $\mathcal{I} \subset \mathcal{B}$ and some $\lambda \in \mathfrak{M}$ satisfying*

$$\lim_{\varepsilon\to 0} \sup \left\{\frac{1}{\lambda I} \mathsf{P}\{\xi I \geqq a\} : I \in \mathcal{I}, |I| < \varepsilon, s \in I^-\right\} = 0 , \quad s \in \mathfrak{S} . \tag{2.12}$$

Proof. Let $\mathcal{I}$ and λ be such as stated, and suppose that $\mathsf{P}\{\xi_a^* \neq 0\} > 0$. Since $\mathcal{I}$ is a covering class, we may then choose some set $I \in \mathcal{I}$ such that $\mathsf{P}\{\xi_a^* I > 0\} > 0$. Letting $\{I_{nj}\} \subset \mathcal{I}$ be a null-array of nested partitions of I, we obtain

$$\mathsf{P}\{\xi_a^* I > 0\} = \mathsf{P} \bigcup_j \{\xi_a^* I_{1j} > 0\} \leqq \sum_j \mathsf{P}\{\xi_a^* I_{1j} > 0\} , \tag{2.13}$$

and we shall prove that this implies

$$\max_j \frac{1}{\lambda I_{1j}} \mathsf{P}\{\xi_a^* I_{1j} > 0\} \geqq \frac{1}{\lambda I} \mathsf{P}\{\xi_a^* I > 0\} . \tag{2.14}$$

In fact, (2.14) follows trivially from (2.13) if $\lambda I = 0$, while if $\lambda I > 0$, insertion of the converse of (2.14) into (2.13) would yield the contradiction

$$\mathsf{P}\{\xi_a^* I > 0\} < \frac{1}{\lambda I} \mathsf{P}\{\xi_a^* I > 0\} \sum_j \lambda I_{1j} = \mathsf{P}\{\xi_a^* I > 0\} .$$

From (2.14) it is seen that there exists some $I_1 \in \{I_{11}, I_{12}, \ldots\}$ satisfying

$$\frac{1}{\lambda I_1} \mathsf{P}\{\xi_a^* I_1 > 0\} \geqq \frac{1}{\lambda I} \mathsf{P}\{\xi_a^* I > 0\} ,$$

and proceeding inductively, we may construct a non-increasing sequence $I_1, I_2, \ldots \in \mathcal{I}$ with $|I_n| \to 0$, such that

$$\frac{1}{\lambda I_n} \mathsf{P}\{\xi I_n \geqq a\} \geqq \frac{1}{\lambda I_n} \mathsf{P}\{\xi_a^* I_n > 0\} \geqq \frac{1}{\lambda I} \mathsf{P}\{\xi_a^* I > 0\} > 0 , \quad n \in \mathbb{N} . \tag{2.15}$$

Since $\{I_n^-\}$ has a non-empty intersection $\{s\}$, (2.15) shows that (2.12) is violated at s. □

Criteria for a.s. diffuseness of random measures are obtained from Theorems 2.5 and 2.6 by letting $a > 0$ be arbitrary. Thus ξ is a.s. diffuse if ξ is a-regular w.r.t. some covering class $\mathcal{I} \subset \mathcal{B}$ for every $a > 0$. In this case we shall say that ξ is *regular* w.r.t. $\mathcal{I}$. Note also that simplicity criteria for point processes may be obtained by taking $a = 2$ in Theorems 2.5 and 2.6. Finally observe that, for point processes, (2.8) is generally true with $a = 1$, since in this case $\xi_1' = 0$ a.s.

We conclude this section with a different kind of simplicity criterion which will play an important role in the sequel.

Lemma 2.7. *Let ξ be a point process on $\mathfrak{S}$ and let $\mathcal{I} \subset \mathcal{B}$ be a DC-semiring. Then ξ is a.s. simple iff*

$$\mathsf{P}\{\xi I > 1\} \leqq \mathsf{P}\{\xi^* I > 1\} , \quad I \in \mathcal{I} . \tag{2.16}$$

Proof. Suppose that (2.16) holds, and conclude that

$$0 \leqq \mathsf{P}\{\xi I < \xi^* I = 1\} = \mathsf{P}\{\xi I > 1\} - \mathsf{P}\{\xi^* I > 1\} \leqq 0 , \quad I \in \mathcal{I} . \tag{2.17}$$

Letting $I \in \mathcal{I}$ be fixed and choosing a null-array $\{I_{nj}\} \subset \mathcal{I}$ of partitions of I, we obtain from (2.17) and Lemma 2.2

$$\begin{aligned}\mathsf{P}\{\xi I > \xi^* I\} &= \mathsf{P} \bigcup_j \{\xi I_{nj} > \xi^* I_{nj}\} \\ &\leqq \mathsf{P} \bigcup_j \{\xi I_{nj} > \xi^* I_{nj} = 1\} + \mathsf{P} \bigcup_j \{\xi^* I_{nj} > 1\} \\ &= \mathsf{P} \bigcup_j \{\xi^* I_{nj} > 1\} = \mathsf{P}\{\max_j \xi^* I_{nj} > 1\} \to 0 ,\end{aligned}$$

which proves that $(\xi - \xi^*) I = 0$ a.s. Since $\mathcal{I}$ is covering, this yields $\xi - \xi^* = 0$ a.s., and so ξ is a.s. simple. The converse assertion is obvious. □

2.3. Absolute continuity

For $p > 1$, let $\|\cdot\|_p$ denote the norm in $L_p(\Omega, \mathcal{A}, \mathsf{P})$, i.e. $\|\eta\|_n = (\mathsf{E}|\eta|^p)^{1/p}$. By Minkowski's inequality, the set function $\|\xi B\|_p$, $B \in \mathcal{B}$, is subadditive for every random measure ξ. Given a null-array $\mathcal{I} = \{I_{nj}\} \subset \mathcal{B}$ of nested partitions of $\mathfrak{S}$, we may hence define the set function

$$\|\xi\|_p I = \lim_{n\to\infty} \sum_j \|\xi I_{nj}\|_p , \quad I \in \mathcal{I} , \tag{2.18}$$

where the summation extends by convention over all j with $I_{nj} \subset I$.

Theorem 2.8. *Fix $p > 1$ and a null-array $\mathcal{I} = \{I_{nj}\} \subset \mathcal{B}$ of nested partitions of $\mathfrak{S}$, and let ξ be a random measure on $\mathfrak{S}$ with $\mu = \mathsf{E}\xi \in \mathfrak{M}$. Then $\xi \ll \mu$ a.s. with a density X*

satisfying $\|X\|_p\,\mu \in \mathfrak{M}$ *iff* $\|\xi\|_p I < \infty$ *for all* $I \in \mathcal{I}$. *In this case,*

$$\|\xi\|_p = \|X\mu\|_p = \|X\|_p\mu .$$

Proof. Let $q > 1$ be such that $p^{-1} + q^{-1} = 1$. When there is no risk for confusion, we shall write $\|\cdot\|$ in place of $\|\cdot\|_p$. For every n, we denote by $I_n(s)$ the set I_{nj} containing s and write $X_n(s) = \xi I_n(s)/\mu I_n(s)$.

Let us first suppose that $\xi = X\mu$. By Fubini's theorem and Hölder's inequality, we get

$$\|\mu X\|_p^p = \mathsf{E}(\mu X)^p = \mathsf{E}(\mu X)\,(\mu X)^{p-1} = \mathsf{E}\mu\big(X(\mu X)^{p-1}\big) = \mu\mathsf{E}\big(X(\mu X)^{p-1}\big)$$
$$\leq \mu(\|X\|_p\,\|(\mu X)^{p-1}\|_q) = (\mu\,\|X\|_p)\,\|\mu X\|_p^{p/q} .$$

If $0 < \|\mu X\|_p < \infty$, we may divide by the second factor on the right to obtain $\|\mu X\| \leq \mu\,\|X\|$. This extends by monotone convergence to the case of infinite $\|\mu X\|$, and for $\|\mu X\| = 0$ our inequality reduces to a triviality. Replacing X by $X \cdot 1_B$ for $B \in \mathcal{B}$ yields $\|\xi B\| \leq (\|X\|\,\mu)\,B$, and it follows by (2.18) that $\|\xi\|\,I \leq (\|X\|\,\mu)\,I$, $I \in \mathcal{I}$.

To prove the converse inequality, note that, for every fixed ω, X_n is a martingale on $(\mathfrak{S}, \mu)$ with a.e. limit X, and conclude by Fubini's theorem that $X_n \to X$ a.e. $\mu \times \mathsf{P}$ (cf. 15.8.3). Applying Fatou's lemma twice and using (2.18), we hence obtain for $I \in \mathcal{I}$

$$(\|X\|\,\mu)\,I = \int_I \|X\|\,d\mu \leq \int_I \liminf_{n\to\infty} \|X_n\|\,d\mu$$
$$\leq \liminf_{n\to\infty} \int_I \|X_n\|\,d\mu = \lim_{n\to\infty} \sum_j \|\xi I_{nj}\| = \|\xi\|\,I .$$

Thus $\|\xi\|\,I = (\|X\|\,\mu)\,I$ holds for all $I \in \mathcal{I}$, which means that the set function $\|\xi\|$ extends (uniquely) to the measure $\|X\|\mu$ on $\mathfrak{S}$. Note in particular that $\|\xi\|\,I < \infty$ for all $I \in \mathcal{I}$ whenever $\|X\|\mu \in \mathfrak{M}$.

It remains to prove that ξ is absolutely continuous whenever the limits in (2.18) are finite. Under this condition, Hölder's inequality yields for any $I \in \mathcal{I}$

$$\mathsf{E}\int_I (X_n)^{1+1/q}\,d\mu = \sum_j \mathsf{E}\xi I_{nj}\left(\frac{\xi I_{nj}}{\mu I_{nj}}\right)^{1/q} \leq \sum_j \|\xi I_{nj}\|_p \left\{\mathsf{E}\left(\frac{\xi I_{nj}}{\mu I_{nj}}\right)\right\}^{1/q}$$
$$= \sum_j \|\xi I_{nj}\|_p \leq \|\xi\|\,I < \infty . \qquad (2.19)$$

Now $\{X_n\}$ is again a martingale on $(\mathfrak{S}, \mu)$ for every fixed ω, so $\{(X_n)^{1+1/q}\}$ is a submartingale, and it follows in particular that the integral $\int_I (X_n)^{1+1/q}\,d\mu$ is non-decreasing in n for every fixed ω. We then obtain from (2.19) by monotone convergence

$$\mathsf{E}\sup_n \int_I (X_n)^{1+1/q}\,d\mu < \infty ,$$

so the inner integral must be a.s. bounded. Thus the martingale $\{X_n\}$ is in fact a.s. uniformly integrable, and in particular it converges in $L_1(I, \mu)$ for every $I \in \mathcal{I}$ (cf. 15.8.2). Writing X for the limit (which is clearly independent of I), we get a.s.

$$\int_I X\,d\mu = \lim_{n\to\infty} \int_I X_n\,d\mu = \sum_j \xi I_{nj} = \xi I , \qquad I \in \mathcal{I} .$$

Thus the measures ξ and $X\mu$ coincide a.s. on $\mathcal{I}$, and so they must agree on $\mathfrak{S}$ also. □

2.4. Exercises

2.1. The measurability of the mapping $\mu \to \mu_d$ as well as of $\mu \to \mu_a^*$ and $\mu \to \mu_a'$, $a > 0$, follows from Lemma 2.3. Give a direct proof based on Lemma 2.2.

2.2. Prove that the events $\{\mu \in \mathfrak{M}$: μ is diffuse$\}$ and $\{\mu \in \mathfrak{N}$: μ is simple$\}$ are measurable. (*Hint*: Consider the measures $\mu - \mu_d$ and $\mu - \mu^*$, respectively.)

2.3. Prove that the notion of null-array of partitions as well as the conditions (2.11) and (2.12) are independent of the choice of metric ϱ.

2.4. Show that, if ξ is a Cox process directed by η, then ξ is a.s. simple iff η is a.s. diffuse, (cf. e.g. Matthes et al. (1978), p. 291). Show also that, if ξ is for some $p > 0$ a p-thinning of η, then ξ and η are simultaneously a.s. simple. (*Hint*: Consider first the case of non-random η.)

2.5. Show that Lemma 2.7 follows directly from Theorem 2.4 in the particular case when $\mathsf{E}\xi_2' \in \mathfrak{M}$. (*Hint*: Apply (2.8) with $a = 2$ and $b = \infty$ to both ξ and ξ^*.)

2.6. Let ξ be a random measure on $\mathfrak{S}$, let $\mathcal{I} \subset \mathcal{B}$ be a DC-semiring and let $a > 0$. Prove that $\xi_a^* = 0$ a.s. iff $\mathsf{P}\{\xi I \geq a\} \leq \mathsf{P}\{\xi_a' I \leq a\}$, $I \in \mathcal{I}$. (*Hint*: Proceed as in the proof of Lemma 2.7.)

2.7. Show by an example that the decomposition in (2.7) is not unique in general, even apart from the order of terms. (*Hint*: We may e.g. take $\mathfrak{S} = \{1, 2\}$, $\nu = 2$, $\beta_1 = 1$, $\beta_2 = 2$, and let $\mathsf{P}\{\tau_1 = 1,\ \tau_2 = 2\} = \mathsf{P}\{\tau_1 = 2,\ \tau_2 = 1\} = 1/2$. A second decomposition is obtained if we replace τ_1 and τ_2 by $\tau_1 \wedge \tau_2$ and $\tau_1 \vee \tau_2$).

2.8. Show that Theorem 2.4 and the converse part of Theorem 2.5 may be false when $\mathsf{E}\xi_a' = \infty$ or $\mathsf{E}\xi = \infty$ respectively. (Cf. Daley (1974) or Matthes et al. (1978), p. 371. *Hint*: Consider (2.8) with $a = 2$ and $b = \infty$, and let ξ be a suitable mixture of the non-random measures $\mu_n \in \mathfrak{N}(\mathbb{R})$, $n \in \mathbb{N}$, where μ_n has unit atoms at $j2^{-n}$, $j \in \mathbb{Z}$. Further define $B_{nj} = ((j-1)\,2^{-n}, j2^{-n}]$.)

2.9. Prove the following stochastic version of Lemma 2.2: Let ξ be a random measure on $\mathfrak{S}$ and let $\mathcal{I} \subset \mathcal{B}$ be a semiring satisfying $\hat{\sigma}(\mathcal{I}) = \mathcal{B}$. Then there exists for every $I \in \mathcal{I}$ some array $\{I_{nj}\} \subset \mathcal{I}$ of finite disjoint partitions of I such that

$$\sum_j 1_{[a,\infty)}(\xi I_{nj}) \xrightarrow{\mathsf{P}} \xi_a^* I \quad \text{as} \quad n \to \infty , \quad a > 0 .$$

(*Hint*: For fixed $a > 0$, let $\{B_{nj}\} \subset \mathcal{B}$ be a null-array of partitions of I and apply Lemma 2.2 to the random measure (ξ_a^*, ξ_a') on the space $2I$ (two identical copies of I). Then use on $2I$ the result of Exercise 1.5 to obtain two approximating partitions $\{U_{nj}\}$ and $\{U_{nj}'\}$ into finite unions of $\mathcal{I}$-sets. Let $\{I_{nj}^{(1)}\} \subset \mathcal{I}$ be a common refinement. In case of k points $a_1, \ldots, a_k \in \mathbb{R}_+$, choose a common refinement $\{I_{nj}^{(k)}\} \subset \mathcal{I}$ of the corresponding sequences. Finally suppose that $a_1, a_2, \ldots$ is a dense sequence in $\mathbb{R}_+'$ containing all fixed jumps of the process $\xi_a^* I$, $a > 0$, and choose for $\{I_{nj}\}$ the "diagonal" sequence in $\{I_{nj}^{(k)}\}$.)

2.10. Use the result of Exercise 2.9 to extend Theorems 2.4 and 2.5, Lemma 2.7 and Exercise 2.6 to the case of semirings $\mathcal{I} \subset \mathcal{B}$ satisfying $\hat{\sigma}(\mathcal{I}) = \mathcal{B}$. (For a direct argument in the point process case, cf. Matthes et al. (1978), p. 34.)

2.11. Prove that Lemma 2.1 remains true for arbitrary σ-finite measures μ on $\mathfrak{S}$, while Lemma 2.2 remains true with $a = 1$ for the subclass of $\mathbb{Z}_+$-valued σ-finite measures.

2.12. Let ξ be a random measure on $\mathfrak{S}$, let $\mathcal{J}$ denote the class of all open or all closed $\mathcal{B}$-sets, and let $a > 0$ and $\lambda \in \mathfrak{M}$. Show that $\xi_a^* = 0$ a.s. if (2.12) holds with I replaced by I^-. (*Hint*: In the case of closed sets, note that $\mathcal{B}_\lambda = \{B \in \mathcal{B}: \lambda \partial B = 0\}$ is a DC-ring (cf. Lemma 4.3 below), and apply Theorem 2.6 to this class. As for the open set case, approximate each $B \in \mathcal{B}_\lambda$ by open $\mathcal{B}$-sets $G \supset B$, cf. 15.6.1.)

2.13. Show that Lemma 2.7 remains true if the family $\{I^\circ, I \in \mathcal{J}\}$ constitutes a base for the topology. (*Hint*: Fix a compact set C and a finite cover by $\mathcal{J}$-sets of diameter $< \varepsilon$.)

2.14. Show that the mapping $(\mu, s) \to \mu\{s\}$ is measurable from $\mathfrak{M} \times \mathfrak{S}$ to R_+. Thus $\xi\{s\}$ is a measurable random process on $\mathfrak{S}$ whenever ξ is a random measure. (*Hint*: Approximate by $\sum_j \mu I_{nj} 1_{I_{nj}}$, where $\{I_{nj}\}$ is a null-array of partitions of $\mathfrak{S}$.)

2.15. Let p, $\{I_{nj}\}$, μ, ξ and X be such as in Theorem 2.8. Prove that

$$\|\xi I_n(s)\|_p / \mu I_n(s) \to \|X\|_p \quad \text{a.e. } (\mu) \text{ and in } L_1(\mu) .$$

(*Hint*: The expression on the left is a submartingale bounded by the martingale $\|\xi\|_p\, I_n(s)/\mu I_n(s)$, hence uniformly integrable with limit $f \leq \|X\|_p$, a.e. and in L_1 (cf. 15.8.1-3). Moreover, its integral converges to that of $\|X\|_p$ by (2.18), so $f = \|X\|_p$ a.e.)

2.16. Fix $p > 1$, and let ξ be a p-th order point process on $\mathfrak{S}$, i.e. such that $\mathsf{E}(\xi B)^p < \infty$, $B \in \mathcal{B}$. Let ξ^p be obtained from ξ by changing every atom size β to β^p. Prove that, for every null-array $\{I_{nj}\} \subset \mathcal{B}$ of nested partitions of $\mathfrak{S}$,

$$\frac{\mathsf{E}(\xi I_n(s))^p}{\mathsf{E}\xi^p I_n(s)} \to 1 \quad \text{a.e. } \mathsf{E}\xi^p .$$

(*Hint*: Check that the left-hand side is an L_1-bounded supermartingale ≥ 1, and that its integral tends to that of 1. Then apply Fatou's lemma.)

2.17. Let ξ be a first order point process on $\mathfrak{S}$, and let $\{I_{nj}\} \subset \mathcal{B}$ be a null-array of nested partitions of $\mathfrak{S}$. Prove that

$$\frac{\mathsf{E}[\xi I_n(s);\, \xi^* I_n(s) = 1]}{\mathsf{E}\xi I_n(s)} \to 1 \quad \text{a.e. } \mathsf{E}\xi .$$

(*Hint*: Check that the left-hand side is a submartingale bounded by 1, hence uniformly integrable, and that its integral tends to that of 1.)

3. Uniqueness

3.1. The general case

Write $\overset{d}{=}$ for equality in distribution, i.e. $\xi \overset{d}{=} \eta$ iff $\mathsf{P}\xi^{-1} = \mathsf{P}\eta^{-1}$.

Theorem 3.1. *Let ξ and η be random measures on $\mathfrak{S}$, and let $\mathcal{J} \subset \mathcal{B}$ be a semiring satisfying $\hat{\sigma}(\mathcal{J}) = \mathcal{B}$. Then the following four statements are equivalent*

(i) $\xi \overset{d}{=} \eta$,

(ii) $\xi f \overset{d}{=} \eta f$, $\quad f \in \mathcal{F}_c$, $\quad$ (ii)$'$ $L_\xi(f) = L_\eta(f)$, $\quad f \in \mathcal{F}_c$,

(iii) $(\xi I_1, \ldots, \xi I_k) \overset{d}{=} (\eta I_1, \ldots, \eta I_k)$, $\quad k \in N$, $\quad I_1, \ldots, I_k \in \mathcal{J}$.

Note in particular that the L-transforms which were calculated in Chapter 1 determine the corresponding distributions uniquely.

Proof. It is obvious that (i) implies (ii), (ii)$'$ and (iii). Suppose conversely that (iii) holds, and define

$$\mathcal{D} = \{M \in \mathcal{M}: \mathsf{P}\{\xi \in M\} = \mathsf{P}\{\eta \in M\}\} .$$

Then $\mathcal{D}$ is clearly closed under proper differences and monotone limits, and it contains $\mathfrak{M}$. Furthermore, it is seen from (iii) that $\mathcal{D}$ contains the class $\mathcal{C}$ of all sets of the form

$$\{\mu \in \mathfrak{M}: \mu I_1 \leq t_1, \ldots, \mu I_k \leq t_k\} , \quad k \in N , \quad I_1, \ldots, I_k \in \mathcal{J} , \quad t_1, \ldots, t_k \in R_+ .$$

Since $\mathcal{C}$ is closed under finite intersections, we may conclude from 15.2.1 that $\mathcal{D} \supset \sigma(\mathcal{C})$. But $\sigma(\mathcal{C}) = \mathcal{M}$ by Lemma 1.4, and so $\mathsf{P}\{\xi \in M\} = \mathsf{P}\{\eta \in M\}$ for all $M \in \mathcal{M}$, i.e. $\xi \overset{d}{=} \eta$.

Let us next assume that (ii)$'$ holds. Then

$$L_{\xi f_1, \ldots, \xi f_k}(t_1, \ldots, t_k) = L_\xi(\sum_j t_j f_j) = L_\eta(\sum_j t_j f_j) = L_{\eta f_1, \ldots, \eta f_k}(t_1, \ldots, t_k)$$

for every $k \in N$, $f_1, \ldots, f_k \in \mathcal{F}_c$ and $t_1, \ldots, t_k \in R_+$, so it follows by 15.5.1 that

$$(\xi f_1, \ldots, \xi f_k) \overset{d}{=} (\eta f_1, \ldots, \eta f_k) , \quad k \in N , \quad f_1, \ldots, f_k \in \mathcal{F}_c .$$

We may now proceed as in the first part of the proof to conclude that (i) holds. Since (ii) trivially implies (ii)$'$, this completes the proof. □

Corollary 3.2. *Let ξ be a Cox process directed by some random measure η on $\mathfrak{S}$. Then the distributions of ξ and η determine each other uniquely. This is also true when ξ is a β-compound of some point process η, provided $\mathsf{P}\beta^{-1}$ is known and such that $\beta \overset{d}{\neq} 0$.*

Proof. If ξ is a Cox process directed by η, we have by Table 1

$$L_\xi(f) = L_\eta(1 - e^{-f}) , \quad f \in \mathcal{F} . \tag{3.1}$$

Writing $1 - e^{-f} = tg$, we may solve for f provided $0 \leq tg < 1$, thus obtaining $f = -\log(1 - tg)$. Hence by (3.1)

$$L_{\eta g}(t) = L_\eta(tg) = L_\xi(-\log(1 - tg)) , \quad 0 \leq t < \|g\|^{-1} , \quad g \in \mathcal{F}_c ,$$

where $||g|| = \sup_t g(t)$, and so it is seen from 15.5.1 that $\mathsf{P}\xi^{-1}$ determines $\mathsf{P}(\eta g)^{-1}$ for all $g \in \mathcal{F}_c$. But by Theorem 3.1, the latter distributions determine $\mathsf{P}\eta^{-1}$.

In case of β-compounds, we get in place of (3.1)

$$L_\xi(f) = L_\eta(-\log L_\beta \circ f)\,, \qquad f \in \mathcal{F}\,. \tag{3.2}$$

where the ring stands for composition of functions. Putting $-\log L_\beta \circ f = tg$ and noting that L_β has a unique inverse L_β^{-1} on the interval $(\mathsf{P}\{\beta = 0\}, 1]$, it is seen that we may solve for f provided $0 < tg < -\log \mathsf{P}\{\beta = 0\}$, thus obtaining $f = L_\beta^{-1} \circ e^{-tg}$. Hence by (3.2)

$$L_{\eta g}(t) = L_\xi(L_\beta^{-1} \circ e^{-tg})\,, \qquad 0 \leq t < -||g||^{-1} \log \mathsf{P}\{\beta = 0\}\,, \qquad g \in \mathcal{F}_c\,,$$

and since $\mathsf{P}\{\beta = 0\} < 1$, the proof may be completed as before. □

3.2. Simplicity and diffuseness

Theorem 3.1 may be partially strengthened as follows.

Theorem 3.3. *Let ξ and η be point processes on $\mathfrak{S}$, and suppose that ξ is a.s. simple. Further suppose that $\mathcal{U} \subset \mathcal{B}$ is a DC-ring while $\mathcal{J} \subset \mathcal{B}$ is a DC-semiring. Then $\xi \overset{d}{=} \eta^*$ iff*

$$\mathsf{P}\{\xi U = 0\} = \mathsf{P}\{\eta U = 0\}\,, \qquad U \in \mathcal{U}\,. \tag{3.3}$$

Furthermore, $\xi \overset{d}{=} \eta$ iff (3.3) holds and in addition

$$\mathsf{P}\{\xi I > 1\} \geq \mathsf{P}\{\eta I > 1\}\,, \qquad I \in \mathcal{J}\,. \tag{3.4}$$

Proof. The necessity assertions are obvious. Suppose conversely that (3.3) holds, and define

$$\mathcal{D} = \{M \in \mathcal{N} \colon \mathsf{P}\{\xi \in M\} = \mathsf{P}\{\eta \in M\}\}\,.$$

Then $\mathcal{D}$ contains $\mathfrak{N}$ and is closed under proper differences and monotone limits. Furthermore, it follows from (3.3) that $\mathcal{D}$ contains the class

$$\mathcal{C} = \{\{\mu \in \mathfrak{N} \colon \mu U = 0\}\,, \quad U \in \mathcal{U}\}\,.$$

Now $\mathcal{C}$ is closed under finite intersections, since $\mathcal{U}$ is closed under finite unions and moreover

$$\{\mu U_1 = 0\} \cap \{\mu U_2 = 0\} = \{\mu(U_1 \cup U_2) = 0\}\,, \qquad U_1, U_2 \in \mathcal{U}\,,$$

so it follows by 15.2.1 that $\mathcal{D} \subset \sigma(\mathcal{C})$. Using Lemma 2.2, it is further seen that the mapping $\mu \to \mu^* I$ is $\sigma(\mathcal{C})$-measurable for every $I \in \mathcal{J}$, and by Lemmas 1.3 and 1.4 this proves that the mapping $\varphi \colon \mu \to \mu^*$ is measurable $\sigma(\mathcal{C}) \to \mathcal{N}$. Using the fact that $\mathcal{D} \supset \sigma(\mathcal{C})$, we thus obtain

$$\mathsf{P}\{\xi^* \in M\} = \mathsf{P}\{\xi \in \varphi^{-1}M\} = \mathsf{P}\{\eta \in \varphi^{-1}M\} = \mathsf{P}\{\eta^* \in M\}\,, \qquad M \in \mathcal{N}\,,$$

which proves that $\xi^* \overset{d}{=} \eta^*$, and hence also that $\xi \overset{d}{=} \eta^*$.

If (3.4) holds in addition, then

$$\mathsf{P}\{\eta^* I > 1\} = \mathsf{P}\{\xi I > 1\} \geq \mathsf{P}\{\eta I > 1\}\,, \qquad I \in \mathcal{J}\,,$$

and if follows by Lemma 2.7 that η is a.s. simple. Hence $\xi \overset{d}{=} \eta$ in this case. □

Theorem 3.4. *Let ξ and η be point processes (or random measures) on $\mathfrak{S}$, and suppose that ξ is a.s. simple (or diffuse respectively). Further suppose that $\mathcal{U} \subset \mathcal{B}$ is a DC-ring while $\mathcal{C} \subset \mathcal{B}$ is a covering class, and that $s, t \in \mathbb{R}$ are fixed with $0 < s < t$. Then $\xi \overset{d}{=} \eta$ iff η a.s. is simple (diffuse) and*

$$\mathsf{E}\, e^{-t\xi U} = \mathsf{E}\, e^{-t\eta U}\,, \qquad U \in \mathcal{U}\,, \tag{3.5}$$

and also iff (3.5) holds and in addition

$$\mathsf{E}\, e^{-s\xi C} \leq \mathsf{E}\, e^{-s\eta C}\,, \qquad C \in \mathcal{C}\,. \tag{3.6}$$

If $\mathsf{E}\xi \in \mathfrak{M}$, then (3.6) may be replaced by the condition

$$\mathsf{E}\,\xi C \geq \mathsf{E}\,\eta C\,, \qquad C \in \mathcal{C}\,. \tag{3.7}$$

It is interesting to observe that (3.3) is formally obtained from (3.5) by letting $t \to \infty$.

Proof. Let us first consider the point process case, and let ξ_p and η_p be p-thinnings of ξ and η respectively. By Table 1,

$$L_{\xi_p}(f) = L_\xi\left(-\log\left[1 - p\left(1 - e^{-f}\right)\right]\right)\,, \qquad f \in \mathcal{F}\,,$$

and putting $f = 1_B$, we get in particular

$$\mathsf{E}\, e^{-u\xi_p B} = \mathsf{E} \exp\left\{\xi B \log\left[1 - p\left(1 - e^{-u}\right)\right]\right\}\,, \qquad B \in \mathcal{B}\,, \quad u \in \mathbb{R}_+\,. \tag{3.8}$$

As $u \to \infty$, we obtain by dominated convergence

$$\mathsf{P}\{\xi_p B = 0\} = \mathsf{E} \exp\left\{\xi B \log\left(1 - p\right)\right\}\,, \qquad B \in \mathcal{B}\,. \tag{3.9}$$

Choosing p such that $\log(1 - p) = -t$, i.e. $p = 1 - e^{-t}$, it follows from (3.5), (3.9) and the corresponding relation for η and η_p that

$$\mathsf{P}\{\xi_p U = 0\} = \mathsf{E}\, e^{-t\xi U} = \mathsf{E}\, e^{-t\eta U} = \mathsf{P}\{\eta_p U = 0\}\,, \qquad U \in \mathcal{U}\,,$$

so Theorem 3.3 yields $\xi_p^* \overset{d}{=} \eta_p^*$. Now ξ and ξ_p are simultaneously a.s. simple, and correspondingly for η and η_p (cf. Exercise 2.4), so we get $\xi_p \overset{d}{=} \eta_p^*$ in general, and if η is a.s. simple, even $\xi_p \overset{d}{=} \eta_p$. In the latter case, Corollary 3.2 yields $\xi \overset{d}{=} \eta$, which proves the first assertion.

In proving the second assertion, we may assume that $\xi_p \overset{d}{=} \eta_p^*$ and that (3.6) holds. Since $0 < s < t$, there exists some $u > 0$ satisfying

$$1 - e^{-s} = (1 - e^{-t})(1 - e^{-u}) = p(1 - e^{-u})\,,$$

or equivalently

$$\log\left[1 - p(1 - e^{-u})\right] = -s\,.$$

Inserting this u into (3.8) and the corresponding relation for η, and using (3.6), we obtain for any $C \in \mathcal{C}$

$$\mathsf{E}\, e^{-u\eta_p^* C} = \mathsf{E}\, e^{-u\xi_p C} = \mathsf{E}\, e^{-s\xi C} \leq \mathsf{E}\, e^{-s\eta C} = \mathsf{E}\, e^{-u\eta_p C}\,,$$

so

$$\mathsf{E}\{e^{-u\eta_p^* C} - e^{-u\eta_p C}\} \leq 0\,, \qquad C \in \mathcal{C}\,.$$

But here the random variable within brackets is non-negative, so it must in fact be a.s. zero, and we obtain $\eta_p^* C = \eta_p C$ a.s., $C \in \mathcal{C}$. Thus it follows as in the proof of Lemma 2.7 that η_p is a.s. simple, and the proof may be completed as before. The last assertion follows by a similar argument based on the relations $\mathsf{E}\xi_p = p\mathsf{E}\xi$ and $\mathsf{E}\eta_p = p\mathsf{E}\eta$ (cf. Table 1).

The random measure case may be proved from Theorem 3.3 by a similar argument, where instead of thinnings we consider Cox processes directed by $t\xi$ and $t\eta$. Alternatively, we may fix an arbitrary $u > t$ and apply the point process case of the present theorem to Cox processes directed by $u\xi$ and $u\eta$. The details are left to the reader. □

3.3. Compound point processes

Theorem 3.5. *Let ξ be a β-compound of some point process η on $\mathfrak{S}$. Further suppose that $p = \mathsf{P}\{\beta > 0\} > 0$, and that $C \in \mathcal{B}$ is such that $C\eta$ is a.s. simple and $\eta C \overset{d}{\neq} 0$. Then $\mathsf{P}\eta^{-1}$ and $\mathsf{P}\beta^{-1}$ are uniquely determined by $\mathsf{P}\xi^{-1}$, p and C.*

Proof. As in (3.8) we get from Table 1

$$\mathsf{E}\, e^{-t\xi B} = \mathsf{E} \exp [\eta B \log L_\beta(t)] , \quad B \in \mathcal{B}, \quad t \in R_+ , \tag{3.10}$$

and since $L_\beta(t) \to \mathsf{P}\{\beta = 0\} = 1 - p$ as $t \to \infty$, it follows by dominated convergence that for arbitrary $B \in \mathcal{B}$

$$\mathsf{P}\{\xi B = 0\} = \begin{cases} \mathsf{E} \exp [\eta B \log (1 - p)] & \text{if } 0 < p < 1 , \\ \mathsf{P}\{\eta B = 0\} & \text{if } p = 1 . \end{cases}$$

Replacing B by $B \cap C$ and noting that $\eta(B \cap C) = (C\eta)B$, it is seen from Theorem 3.4 or 3.3 respectively that $\mathsf{P}(C\eta)^{-1}$ and hence also $\mathsf{P}(\eta C)^{-1}$ is uniquely determined. Letting $B = C$ in (3.10), we further obtain

$$L_{\xi C} = L_{\eta C} \circ (-\log L_\beta) ,$$

and since $\eta C \overset{d}{\neq} 0$ by assumption, $L_{\eta C}$ has a unique inverse $L_{\eta C}^{-1}$ on the interval $(\mathsf{P}\{\eta C = 0\}, 1]$, and we get

$$L_\beta = \exp \{-L_{\eta C}^{-1} \circ L_{\xi C}\} .$$

By 15.5.1. this implies that even $\mathsf{P}\beta^{-1}$ is unique, and so we may apply Corollary 3.2 to complete the proof. □

3.4. Exercises

3.1. Show that Theorem 3.1 remains true with $\mathcal{F}_c$ in (ii) replaced by the class of simple functions over $\mathcal{J}$. (Cf. Exercise 1.1 for a definition.)

3.2. Show that Theorem 3.3 (and hence also Theorem 3.4) remains true for any ring $\mathcal{U} \subset \mathcal{B}$ and semiring $\mathcal{J} \subset \mathcal{B}$ satisfying $\hat{\sigma}(\mathcal{U}) = \hat{\sigma}(\mathcal{J}) = \mathcal{B}$. (*Hint*: Use 15.2.2 to extend (3.3) to $\mathcal{B}$, and then apply the result of Exercise 2.10. Cf. Matthes et al. (1978), pp. 31, 34.)

3.3. Apply the method involving thinnings and Cox processes to the second assertion in Theorem 3.3 to obtain an alternative to condition (3.6) in Theorem 3.4.

3.4. Let $\mathcal{C} \subset \mathcal{B}$ be a covering class and let $s > 0$ be fixed. Show that Theorem 3.3 remains true with (3.4) replaced by (3.6) or (3.7).

3.5. Let ξ, η, $\mathcal{U}$ and t be such as in Theorem 3.4, and suppose that (3.5) holds. Show that, in the point process case,

$$L_\eta(f) \leqq L_\xi(f) \leqq L_{\eta^*}(f) , \quad f \in \mathcal{F}, \quad \|f\| \leqq t ,$$

while in the random measure case,

$$L_\eta(f) \leqq L_\xi(f) \leqq L_{\eta_d}(f) , \quad f \in \mathcal{F} , \quad \|f\| \leqq t .$$

3.6. Prove that, if the class $\mathcal{C}$ in Theorem 3.4 is a DC-semiring, then ξ may be allowed to have atoms of fixed size and location. (Cf. the proof of Lemma 7.10 below.)

3.7. Assume that $\mathfrak{S}$ is countable. Show that there exists some fixed function $f \in \mathcal{F}$ such that $\mathsf{P}\xi^{-1}$ is determined by $\mathsf{P}(\xi f)^{-1}$ for any point process ξ on $\mathfrak{S}$ with $\xi\mathfrak{S} < \infty$ a.s. (*Hint*: Let the numbers $f(s)$, $s \in \mathfrak{S}$, be rationally independent and bounded above and below by positive constants.)

3.8. Show that the distribution of a point process ξ on $\mathfrak{S}$ is not, in general, determined by $\mathsf{P}(\xi B)^{-1}$ for all $B \in \mathcal{B}$. (Cf. Lee (1968) and Matthes et al. (1978), p. 18. *Hint*: It is enough to take $\mathfrak{S} = \{1, 2\}$ and assume that $\xi_3^* = 0$.)

3.9. It is essential in Theorem 3.3 that $\mathcal{U}$ be a ring. In fact, show that if $\mathcal{J}$ is the class of real intervals, then the distribution of a simple point process ξ on R is not even determined, in general, by $\mathsf{P}(\xi I)^{-1}$, $I \in \mathcal{J}$. (Cf. Lee (1968), Moran (1967) and Goldman (1967), as well as Matthes et al. (1978), p. 372. *Hint*: For arbitrary $n \in N$, let ξ be a simple point process on $\{1, \ldots, n\}$. If n is sufficiently large ($\geqq 6$), the $2^n - 1$ independent parameters in $\mathsf{P}\xi^{-1}$ cannot be determined by $\sum_{k=1}^{n} k(n - k + 1) \leqq n^3$ linear conditions.)

3.10. Use the method of Exercise 3.9 to show that the distribution of a simple point process on R is not even determined by all $\mathsf{P}(\xi I_1, \ldots, \xi I_k)^{-1}$, $I_1, \ldots, I_k \in \mathcal{J}$, with $k \in N$ fixed. (Cf. Szász (1970).)

3.11. Suppose that the ξ_ϑ in Lemma 1.7 are simple point processes, and let $\mathcal{U} \subset \mathcal{B}$ be a DC-ring. Show that the mixture of $\mathsf{P}\xi_\vartheta^{-1}$ exists iff $\mathsf{P}\{\xi_\vartheta U = 0\}$ is $\mathcal{T}$-measurable for every $U \in \mathcal{U}$. Prove an analogous result related to Theorem 3.4.

3.12. Let ξ be a random measure on $\mathfrak{S}$. Show that ξ has independent increments iff $B_1\xi, \ldots, B_k\xi$ are independent for any $k \in N$ and disjoint $B_1, \ldots, B_k \in \mathcal{B}$. (*Hint*: Consider arbitrary partitions of $B_1, \ldots, B_k$ and apply 15.2.1 to each component. Cf. Matthes et al. (1978), p. 17.)

3.13. Let $\mathcal{U} = \mathcal{C} \subset \mathcal{B}$ be a DC-ring. Show that the first assertion in Theorem 3.3 and the first two assertions in Theorem 3.4 remain true for random elements in $(\mathfrak{M}_\mathcal{U}, \mathcal{M}_\mathcal{U})$, as defined in Exercise 1.8. (Cf. Exercise 2.11.)

3.14. Let $p \in (0, 1]$, and let ξ be a p-thinning of η. Show that ξ and η are simultaneously Cox. (*Hint*: Use Corollary 3.2.)

3.15. Let ξ and η be simple point processes (or diffuse random measures) on $\mathfrak{S}$, let $\mathcal{U} \subset \mathcal{B}$ be a DC-ring and let $t \in (0, \infty]$ (or $t \in (0, \infty)$ respectively) be fixed. Show that ξ and η are independent iff

$$\mathsf{E}\, e^{-t(\xi U + \eta V)} = \mathsf{E}\, e^{-t\xi U}\, \mathsf{E}\, e^{-t\eta V} , \quad U, V \in \mathcal{U}.$$

(*Hint*: Proceed as in the proofs of Theorems 3.3 and 3.4.)

3.16. Let ξ be a Cox process based on η, and let $f: \mathfrak{S} \to \mathfrak{S}'$ be such as in Exercise 1.9. Show that ξf^{-1} is then a Cox process based on ηf^{-1}. State and prove the corresponding assertion for compound point processes and for randomizations. (*Hint*: Calculate L-transforms and use Theorem 3.1.)

3.17. Show that a simple point process ξ is Poisson iff ξB is a Poisson variable for every $B \in \mathcal{B}$. (*Hint*: Let η be a Poisson process with intensity $\mathsf{E}\xi$, and apply Theorem 3.3. Cf. Matthes et al. (1978), p. 58.)

3.18. It is enough to assume in Theorem 3.3 that $\mathcal{J}$ is such as in Exercise 2.13. (For a further extension of Theorem 3.3, see Exercise 4.13 below.)