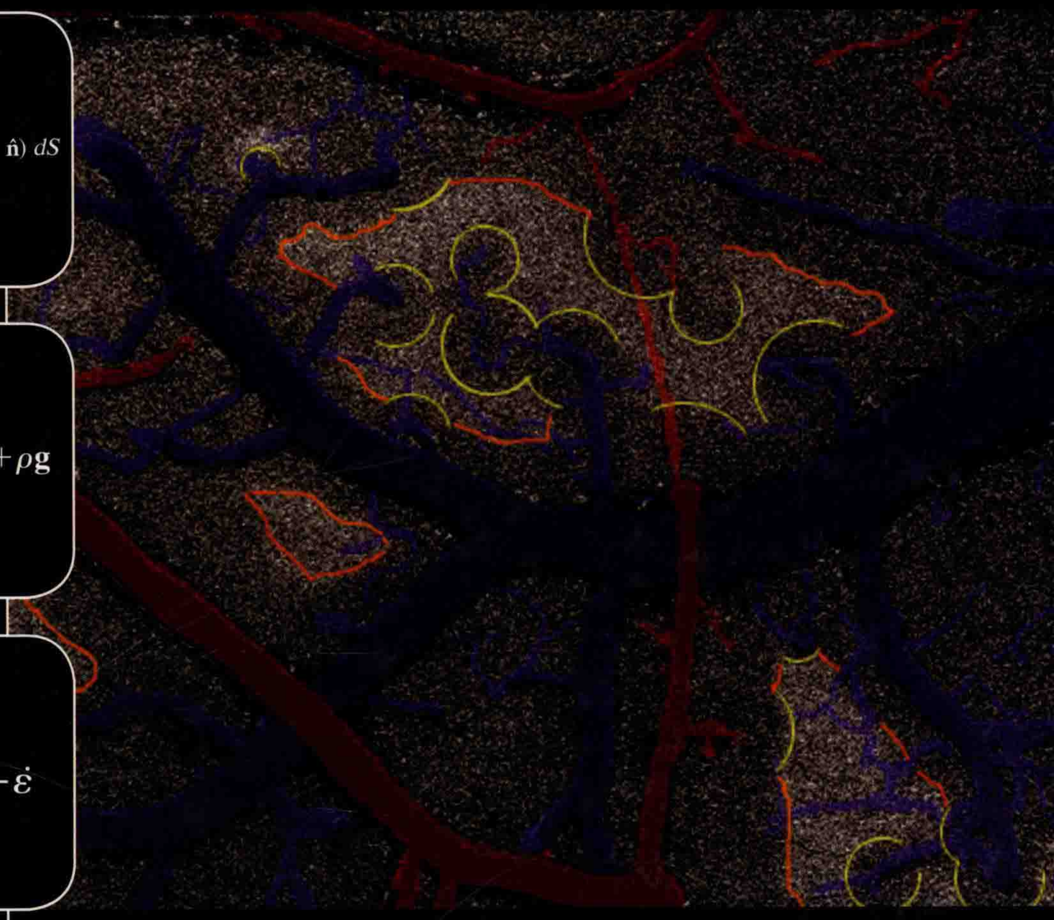
CAMBRIDGE TEXTS IN BIOMEDICAL ENGINEERING

Problems for Biomedical Fluid Mechanics and Transport Phenomena

$$0 = \frac{d}{dt}\int_{CV} \rho \, dV + \int_{CS} \rho(\mathbf{v}\cdot\hat{\mathbf{n}})\, dS$$

$$\rho \frac{\mathcal{D}\mathbf{v}}{\mathcal{D}t} = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}$$

$$\frac{\mathcal{D}C}{\mathcal{D}t} = \mathrm{D}\nabla^2 C + \dot{\varepsilon}$$



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Problems for Biomedical Fluid Mechanics and Transport Phenomena

How does one deal with a moving control volume? What is the best way to make a complex biological transport problem tractable? Which principles need to be applied to solve a given problem? How do you know whether your answer makes sense?

This unique resource provides over 200 well-tested biomedical engineering problems that can be used as classroom and homework assignments, quiz material, and exam questions. Questions are drawn from a wide range of topics, covering fluid mechanics, mass transfer, and heat transfer applications. These problems, which are motivated by the philosophy that mastery of biotransport is learned by practice, will aid students in developing the key skills of determining which principles to apply and how to apply them.

Each chapter starts with basic problems and progresses to more difficult questions. Lists of material properties, governing equations, and charts provided in the appendices make this book a fully self-contained resource. Solutions to problems are provided online for instructors.

Mark Johnson is Professor of Biomedical Engineering, Mechanical Engineering, and Ophthalmology at Northwestern University. He has made substantial contributions to the study of the pathogenesis of glaucoma and of age-related macular degeneration of the retina. His academic interests include biofluid and biotransport issues, especially those related to ocular biomechanics.

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CAMBRIDGE TEXTS IN BIOMEDICAL ENGINEERING

Cambridge Texts in Biomedical Engineering provide a forum for high-quality textbooks targeted at undergraduate and graduate courses in biomedical engineering. It covers a broad range of biomedical engineering topics from introductory texts to advanced topics, including biomechanics, physiology, biomedical instrumentation, imaging, signals and systems, cell engineering, and bioinformatics, as well as other relevant subjects, with a blending of theory and practice. While aiming primarily at biomedical engineering students, this series is also suitable for courses in broader disciplines in engineering, the life sciences and medicine.

To my wife, son, parents, and family,
and to my mentors, colleagues, and students,
who have all in their own ways contributed to this book
Mark Johnson

To my students, colleagues, and family,
who have all taught me so much.
C. Ross Ethier

"A tremendously valuable resource for bioengineering students and instructors that contains problems scaling from the molecular to whole body level. Nearly every system in the body is included, as well as a variety of clinically and industrially relevant situations. The problems are aimed at instruction in applying basic physical principles in a variety of settings, and include entertaining topics such as squid swimming, elephant ear heat transfer, whistling to spread germs, and air friction over a bicyclist. What fun!"

James E. Moore Jr., Imperial College London

"The problems and solutions represent an invaluable resource for instructors. In addition, the step-by-step procedure described in section 1.3 is a wonderfully insightful reminder of what students really need to know to be successful in solving fluid mechanics problems. Instructors would do well to teach this procedure at the beginning and to refer to it consistently throughout the course."

M. Keith Sharp, University of Louisville

"A book devoted solely to biologically relevant problems in fluid mechanics and transport is a very welcome addition to the teaching armamentarium in this area. Problems related to cardiovascular, respiratory and ocular physiology are emphasized, deriving from the substantial research expertise of the authors. The problems are very interesting and in many cases very challenging. They cover a range of difficulty that should be appropriate for both undergraduate and graduate level courses and more than enough topics to provide substantial breadth. Overall excellent! Now I'm looking forward to working out my own solutions and maybe peeking at the solution manual."

John M. Tarbell, The City College of New York

Preface

This book arose out of a need that frequently faced us, namely coming up with problems to use as homework in our classes and to use for quizzes. We have found that many otherwise excellent textbooks in transport phenomena are deficient in providing challenging but basic problems that teach the students to apply transport principles and learn the crucial engineering skill of problem solving. A related challenge is to find such problems that are relevant to biomedical engineering students.

The problems included here arise from roughly the last 20–30 years of our collective teaching experiences. Several of our problems have an ancestry in a basic set of fluid mechanics problems first written by Ascher Shapiro at MIT and later extended by Ain Sonin, also at MIT. Roger Kamm at MIT also generously donated some of his problems that are particularly relevant to biomedical transport phenomena. Thanks are due to Zdravka Cankova and Nirajan Rajkarnikar, who helped with proof-reading of the text and provided solutions for many of the problems.

For the most part, the problems in this book do not involve detailed mathematics or theoretical derivations. Nor do they involve picking a formula to use and then plugging in numbers to find an answer. Instead, most of the problems presented require skills in problem solving. That is, much of the challenge in these problems involves deciding how to approach them and what principle or principles to apply.

Students will need to understand how to pick a control volume, and that multiple control volumes are necessary for some problems. How does one deal with a moving control volume? How many principles need to be applied to solve a given problem? How do you know whether your answer makes sense? Students who are struggling or stuck on a particular problem will want to know how they should proceed in such cases. The problems presented here will raise all of these issues for students.

In the first chapter, we give general principles of problem solving, and present the Reynolds transport theorem. We also show an example of how we would approach and solve one problem. However, problem solving is best learned by doing problems. Seeing someone else solve a problem is not nearly as educational. We hope that we have provided a wide variety of problems in different areas of transport phenomena, most at the basic level, that aids in the development of problem-solving skills for students in these areas. Each chapter of problems is organized such that the easier problems are at the beginning of the chapter, and then the problems become progressively harder. The exception to this rule is that heat transfer problems are to be found at the end of each chapter.

Contents

1 Problem solving

In this introductory chapter, we begin with a derivation of the Reynolds transport theorem, which is central to conservation principles applied to control volumes. Then, we turn to the issue of how to approach problem solving.

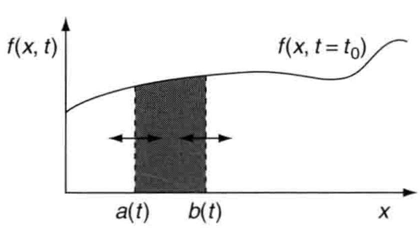
1.1 The Reynolds transport theorem

Quantities, such as mass, momentum, energy and even entropy and money, are conserved in the sense that the following principle can be applied to a system.

Input + Generation = Output + Accumulation

The system normally considered in transport phenomena for application of this principle is a control volume. The equation makes intuitive sense and is simple to apply in many cases. However, when moving control volumes and reference frames are examined, or when transport of quantities that have direction (such as momentum) is considered, intuition is less reliable. We here derive a rigorous version of this conservation principle and, in the process, discover the wide applicability of the Reynolds transport theorem. We note that more intuitive formulations of this principle can be found in other texts (e.g. *Fluid Mechanics* by Potter and Foss).

We consider a generalization of Leibniz's rule for the differentiation of integrals. Consider a given function[1] $f(x)$ and the definite integral (M) of this function between $x = a$ and $x = b$. Let both this function and the limits of integration be functions of time (t) (see the figure):



[1] Note that f could be either a scalar- or a vector-valued function. We write it here as a scalar (unbolded).

$$M = \int_{a(t)}^{b(t)} f(x,t)dx$$

Using the chain rule, we can find how the value of this integral changes with time:

$$\frac{dM}{dt} = \frac{\partial M}{\partial t} + \frac{\partial M}{\partial a}\frac{da}{dt} + \frac{\partial M}{\partial b}\frac{db}{dt}$$

or

$$\frac{dM}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t} dt + f[b(t),t]\frac{db}{dt} - f[a(t),t]\frac{da}{dt}$$

This is Leibniz's rule, which is well known from calculus. M changes with time not only due to temporal changes in f, but also because the boundaries of integration move. Note that the temporal derivative was taken inside the integral, since a and b are held constant in the partial derivative. This will be important when we consider moving control volumes.

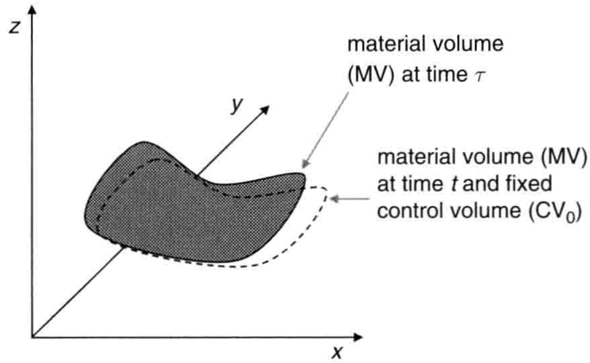
We now look to apply a similar principle but in three dimensions, relating the time rate of change of a moving system to that of a stationary system. This is particularly important in transport phenomena, not simply because our systems are frequently moving, but, more importantly, because our laws of physics are derived for material volumes, not control volumes.

A material volume is a fixed, identifiable set of matter.[2] A control volume is a region of space, fixed or moving, that we choose to analyze. Our laws of physics apply directly to matter, not to control volumes. For example, physics tells us that (for non-relativistic systems) mass is conserved. Thus, the mass of a given material volume is always constant. But the mass in a control volume can change.

Solving a problem by tracking the moving material volume is known as a Langrangian approach. It is typically quite difficult to solve problems in this way since material volumes change their location and shape due to their motion. Analysis is facilitated by use of a control volume whose shape and motion can be specified; such an approach is known as Eulerian. However, to use an Eulerian approach, we require the Reynolds transport theorem, which allows us to relate physical laws that are derived for material volumes to a principle that applies to

[2] Also referred to by some authors as a "control mass." Note that the use of the term "fixed" in the above definition does not imply that the material volume is not moving; rather, it means that its constituent parts are neither destroyed nor created, although they can be transformed into other components through e.g. chemical reactions.

control volumes. In other words, the Reynolds transport theorem acts as a "bridge" between material volumes, where the physical laws are defined, and control volumes, which are more convenient for analysis.



Consider a moving material volume as shown in the figure above. This material volume is moving such that it occupies the region surrounded by the dashed line at time t and the solid line at a later time τ. Note that the points within the material volume are not all necessarily moving with the same velocity (e.g. a fluid or a deforming solid).

Pick a control volume that coincides with the material volume at time t. We define M as the integral of a function $f(\boldsymbol{x}, t)$ over the material volume,

$$M = \int_{\mathrm{MV}} f(\boldsymbol{x}, t) d\boldsymbol{x}$$

where $\boldsymbol{x} = (x, y, z)$. We will relate M to the integral of the same function, $f(\boldsymbol{x}, t)$, over the control volume.

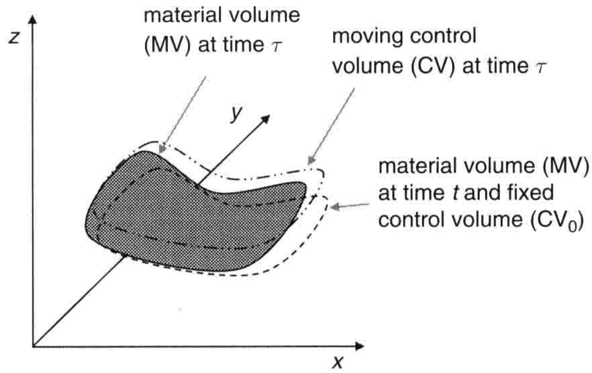
We use an analogous approach to that leading to Leibniz's equation. We consider the integral of a function $f(\boldsymbol{x}, t)$ over the material volume. M changes with time due both to temporal changes in $f(\boldsymbol{x}, t)$ and to the motion of the boundary of the domain of integration. Noting that the final two terms in Leibniz's equation arise due to the flux of f at the boundary carried by the material's velocity out of the control volume (and thus normal to the control surface), we find that the three-dimensional equivalent of Leibniz's equation becomes

$$\frac{dM}{dt} = \int_{\mathrm{CV}_0} \frac{\partial f(\boldsymbol{x}, t)}{\partial t} d\boldsymbol{x} + \int_{\mathrm{CS}_0} f(\boldsymbol{x}, t) \left(\vec{V}_{\mathrm{MV}} \cdot \hat{n} \right) dS$$

where CS_0 is the surface surrounding the control volume CV_0, $\vec{V}_{MV}$ is the velocity of the material volume, and $\hat{n}$ is the outward pointing unit normal.

This is the Reynolds transport theorem for a stationary control volume. It relates the time rate of change of an intensive function, f (a parameter per unit volume), integrated over a material volume to the integral of that intensive function integrated over a control volume. The second integral over the control surface involves the flux of material entering or leaving the control volume. Note that no material enters or leaves a material volume (by definition).

It is frequently convenient when solving transport problems to consider moving control volumes. To generalize the Reynolds transport theorem, consider both a stationary control volume CV_0 and a control volume CV moving at velocity $\vec{V}_{CV}$, and their respective surfaces, CS_0 and CS (see the figure below).



Now, to find the generalized Reynolds transport theorem for the moving control volume, we use the Reynolds transport theorem twice: the first time relating the moving material volume to the stationary control volume, and the second time relating the moving control volume to the stationary control volume:

$$\frac{d}{dt}\int_{MV} f(\boldsymbol{x},t)d\boldsymbol{x} = \int_{CV_0} \frac{\partial f(\boldsymbol{x},t)}{\partial t} d\boldsymbol{x} + \int_{CS_0} f(\boldsymbol{x},t)\left(\vec{V}_{MV}\cdot\hat{n}\right)dS$$

$$\frac{d}{dt}\int_{CV} f(\boldsymbol{x},t)d\boldsymbol{x} = \int_{CV_0} \frac{\partial f(\boldsymbol{x},t)}{\partial t} d\boldsymbol{x} + \int_{CS_0} f(\boldsymbol{x},t)\left(\vec{V}_{CV}\cdot\hat{n}\right)dS$$

On subtracting the second equation from the first, rearranging, and evaluating at time t when the two control volumes are coincident (so that CS and CS_0 are identical), we find

$$\frac{d}{dt}\int_{\mathrm{MV}} f(\boldsymbol{x},t)d\boldsymbol{x} = \frac{d}{dt}\int_{\mathrm{CV}} f(\boldsymbol{x},t)d\boldsymbol{x} + \int_{\mathrm{CS}} f(\boldsymbol{x},t)\left(\vec{V}_{\mathrm{rel}}\cdot\hat{n}\right)dS$$

where $\vec{V}_{\mathrm{rel}}$ is the velocity of the material volume relative to the moving control volume. This is the general form of the Reynolds transport theorem, and it is valid for stationary and moving control volumes.

The physical interpretation of this equation is useful. This is a conservation law for any conserved quantity f, in which f is an intensive variable (expressed per unit volume). The term on the left-hand side of the equation is the rate at which f is generated. The first term on the right-hand side of the equation is the accumulation term: the rate at which f accumulates in the control volume. The final term is the flux term, characterizing the balance of the flux of f out of and into the control volume due to flow. Thus, the Reynolds transport theorem recovers our initial conservation principle, namely

Generation = Accumulation + Output – Input

1.2 Application of the Reynolds transport theorem

By applying the laws of physics to the left-hand side of this equation, conservation laws that apply to control volumes can be generated. For example, when considering mass conservation, the function f becomes the fluid density ρ (mass per unit volume, an intensive variable). Then the left-hand side of the equation is simply the time rate of change of the mass of the material volume. Since this mass is constant (mass is not generated), we find that

$$0 = \frac{d}{dt}\int_{\mathrm{CV}} \rho(\boldsymbol{x},t)d\boldsymbol{x} + \int_{\mathrm{CS}} \rho(\boldsymbol{x},t)\left(\vec{V}_{\mathrm{rel}}\cdot\hat{n}\right)dS$$

This is the mass-conservation equation, which is valid for all non-relativistic control volumes, indicating that accumulation in a control volume results from an imbalance between the influx and outflow of mass from a control volume.

For species conservation, we let $f = C_i$ (moles of species i per unit volume). There are two important differences from the law of mass conservation. First, there is the possibility of generation or destruction of species i due to chemical reactions. We will let the net generation rate of species i be Ψ_i, i.e. the production rate minus the destruction rate. Second, in addition to the flow carrying species i ($C_i\vec{V}_{\mathrm{rel}}$), the diffusion of this species needs to be accounted for.

The diffusional flux of species i is given by Fick's law of diffusion[3]: $\vec{j_i} = -D_i\nabla C_i$, where D_i is the diffusion coefficient of species i. Taking the dot

[3] For isothermal, isobaric conditions.

product of this vector with the unit outward normal to the control surface and integrating over the control surface gives the total net diffusional transport out of the control volume. We then use the Reynolds transport theorem to find the species conservation equation:

$$\int_{\mathrm{CV}} \Psi_i(\boldsymbol{x},t)d\boldsymbol{x} = \frac{d}{dt}\int_{\mathrm{CV}} C_i(\boldsymbol{x},t)d\boldsymbol{x} + \int_{\mathrm{CS}} C_i(\boldsymbol{x},t)\left(\vec{V}_{\mathrm{rel}}\cdot\hat{n}\right)dS + \int_{\mathrm{CS}}(\vec{J}_i(\boldsymbol{x},t)\cdot\hat{n})dS$$

Likewise, if we allow that $f = \rho\vec{V}$ (momentum per unit volume, a vector), then the left-hand side of the Reynolds transport theorem is the time rate of change of the momentum of the material volume. This we know from Newton's second law must be the sum of the forces acting on the material volume. Thus, the momentum equation is derived:

$$\sum\vec{F} = \frac{d}{dt}\int_{\mathrm{CV}}\rho\vec{V}(\boldsymbol{x},t)d\boldsymbol{x} + \int_{\mathrm{CS}}\rho\vec{V}(\boldsymbol{x},t)\left(\vec{V}_{\mathrm{rel}}\cdot\hat{n}\right)dS$$

This is a vector equation that describes a momentum balance in each of the coordinate directions.

Note that we have imposed no restrictions on the motion of our control volume when deriving the momentum-conservation equation. It can even be accelerating. However, the reference frame (which is not the same as the control volume) cannot be accelerating because Newton's second law does not hold (without modification) for non-inertial reference frames.

Note also that the second integral in the above equation contains two velocities that are not necessarily the same. One is the velocity of the material volume (the fluid), while the other is the relative velocity between the fluid and the control volume. The velocities can even be in different directions (e.g. transferring x-momentum in the y-direction such as might occur when one skater passes another and throws a book in a perpendicular direction that is caught by the slower skater).

The relative velocity in the last term of the momentum equation is present as the dot product with the outward normal, so only the component of $\vec{V}_{\mathrm{rel}}$ that carries material across the control surface contributes to the integral. The sign of a term can be confusing to determine: the sign of any component of $\vec{V}$ is established by the coordinate direction, e.g. a positive V_x is one that points in the same direction as the x-axis. However, the sign of the term $\vec{V}_{\mathrm{rel}}\cdot\hat{n}$ is determined only by whether fluid is entering or leaving the control volume, being negative or positive, respectively. Students must pay attention to this tricky point!