de Gruyter
Studies in Mathematics
1

Wilhelm P. A. Klingenberg

Riemannian Geometry

2nd Edition

Wilhelm P. A. K

Riemannian Geometry

Second Revised Edition

W DE G

Walter de Gruyter
Berlin · New York 1995

Author
Wilhelm P. A. Klingenberg
Mathematisches Institut
der Universität Bonn
Wegelerstraße 10
D-53115 Bonn

For my wife

1991 Mathematics Subject Classification: 53-02; 53Axx, 53Bxx, 53Cxx, 49Lxx, 49Rxx, 57Rxx, 58Axx, 58Bxx, 58Cxx, 58Dxx, 58Exx, 58Fxx

Keywords: Differential Geometry, manifolds, global analysis, analysis on manifolds, calculus of variations

♾ Printed on acid-free paper which falls within the guidelines of the ANSI to ensure permanence and durability.

Library of Congress Cataloging-in-Publication Data

Klingenberg, Wilhelm P. A., 1924–
Riemannian geometry / Wilhelm Klingenberg. – 2nd rev. ed.
p. cm. – (De Gruyter studies in mathematics ; 1)
Includes bibliographical references (p. –) and index.
ISBN 3-11-014593-6 (acid-free)
1. Geometry, Riemannian. I. Title. II. Series.
QA649.K544 1995
516.3′73–dc20 95-2589

Die Deutsche Bibliothek – CIP-Einheitsaufnahme

Klingenberg, Wilhelm:
Riemannian geometry / Wilhelm P. A. Klingenberg. – 2., rev. ed. –
Berlin ; New York : de Gruyter, 1995
(De Gruyter studies in mathematics ; 1)
ISBN 3-11-014593-6
NE: GT

Printed in Germany.
Typesetting and Printing: Fa. Tutte, Salzweg.
Binding: Dieter Mikolai, Berlin. Cover design: Rudolf Hübler, Berlin.

Preface to the Second Edition

It was with great satisfaction that I accepted the invitation of the publisher to prepare a Second Edition of my *Riemannian Geometry*. Since the book first appeared, numerous misprints and several errors came to my attention. At an age where printing with the assistance of a computer has become quite simple, it is somewhat paradoxical that at the same time it became very time consuming and costly to make changes in a published text. I highly appreciate that nevertheless the Walter de Gruyter Company was willing to take into account my suggestions in an oldfashioned way. In particular, I want to thank Dr. M. Karbe, who supported me through all stages.

Since my *Riemannian Geometry* first appeared in 1983, the field has experienced a tremendous growth and extension. In the Second Eddition, I only sporadically can hint at some of the high lights. Fortunately, there exists an excellent survey of the state of affairs at the beginning of the Nineties: It is part 3 of volume 54, dedicated to Riemannian Geometry, in the Proc. Symp. Pure Math., Providence, RI: Amer. Math. Soc. 1993.

However, there is one result of very recent origin, which I am quite happy to include into the Second Edition. I call it the Main Theorem for Surfaces of Genus 0 and it states: There always exist on such a surface infinitely many geometrically distinct closed geodesics.

It is a truly centennial result since, already in 1905, H. Poincaré showed that on a convex surface there exists a simple closed geodesic. His work was continued and extended by G. Birkhoff, L. Lusternik, L. Schnirelmann and many others, until finally, in 1993, the combined efforts of V. Bangert and J. Franks led to a proof of the Main Theorem in full generality. In the same year, N. Hingston published a paper which only uses methods which have been developed in this monograph and avoids the approach that Franks used. It therefore became quite natural for me to present in the final section of chapter 3 a complete proof of the Main Theorem. I feel that this constitutes an important and beautiful finale to my work.

Bonn, Januar 1995 Wilhelm P. A. Klingenberg

Preface

The present book is an outcome of my course on Riemannian Geometry. Its origin can be traced back to a series of special lectures which I gave during the summer semester 1961 in Bonn. At that time, D. Gromoll and W. Meyer were among my students and in 1967 we jointly published in the Lecture Notes Series our "Riemannsche Geometrie im Großen".

These lectures have met with great interest, because for the first time a concise introduction into Riemannian Geometry was combined with global methods culminating in the so-called Sphere Theorem, which states that the underlying topological manifold of a simply connected Riemannian manifold with suitably restricted positive curvature is a sphere.

Over the past twenty years, global Riemannian Geometry has experienced considerable growth in various areas. Here I wish to mention in particular the work of Gromov [1], [2], [3] on manifolds with restrictions on the curvature and the numerous results on the eigenvalues of the Laplace operator. Also – and this is the field in which I have been most active myself – a great number of new results on the existence and on the properties of closed geodesics have been obtained. For an excellent survey of the present state of research cf. Yau [1].

It is only natural that in my course I have chosen topics which are close to my own areas of research. But I have always begun with a full exposition of the classical, local Riemannian Geometry.

Thus, chapter 1 is devoted to the foundations. What is unusual here is that from the very beginning I have allowed manifolds to be modelled on separable Hilbert spaces. This presents no difficulties when compared with the case of finite dimensional manifolds and it has the advantage of yielding the necessary framework for later applications in chapter 2. Of course, there are some differences between Hilbert manifolds and finite dimensional manifolds, which appear for the first time when considering the tensor product. However, for most of the basic results, the step from finite dimensions to Hilbert space is no bigger than the step from 2 dimensions to n dimensions. Whenever the restriction to the finite dimensional case brings about some simplification, I have pointed this out clearly.

In chapter 1 – and the same is true for the later chapters – the first part of a section is usually more basic than the rest. At least this is the case when the sections are longer than 10 pages or so. While the expert will have no difficulty in constructing a 'basic course' from our material, the beginner should keep in mind that it might be wise to switch to the next section when he reaches subjects like general vector bundles, submersions or focal points, to mention a few. He can always return to the previous sections when the need arises.

In chapter 2, entitled *Curvature and Topology*, I restrict myself to finite dimensional manifolds because the local compactness of the manifold is needed. Complete manifolds are studied and there follows a rather complete account of the theory of symmetric spaces from the point of view of Riemannian Geometry which differs from the usual, more algebraic approach. After this, I develop in three sections the basic theory of the manifold of curves. Here, I take advantage of the fact that in chapter 1 Riemannian manifolds modelled on Hilbert Spaces were allowed.

When it comes to the critical point theory I only develop the Lusternik-Schnirelmann approach. I do not enter into the much more delicate Morse Theory. The full power of this approach has not been sufficiently recognized. Among other things I show that one can give a completely elementary proof of the fundamental estimate on the injectivity radius of 1/4-pinched manifolds.

The chapter continues with a simplified proof of the Alexandrov-Toponogov Comparison Theorem. This is an essential tool in the proof of the Sphere Theorem, given in the next section. I conclude with the basic constructions on non-compact manifolds of positive curvature.

With the end of chapter 2 I have covered all the material contained in the "Riemannsche Geometrie im Großen" and indeed much more; e.g., symmetric spaces and the manifold of curves with its various important submanifolds.

In chapter 3, entitled *Structure of the Geodesic Flow*, I deal with a subject which, traditionally, is not presented in a course on Riemannian Geometry. I feel, however, that this field should not be left to specialists in ergodic theory or Hamiltonian systems. Rather, it should be tied more closely to Riemannian Geometry proper. In fact, it is one of the oldest fields of research. Thus, e.g., the geodesic flow on the ellipsoid was even studied by C. G. J. Jacobi and the problem of stability for periodic orbits plays a fundamental rôle in Poincaré's investigations on Celestial Mechanics. I present many of the classical results together with numerous examples. Among them, there are theorems for periodic orbits with elementary proofs employing only the Lusternik-Schnirelmann theory developed in chapter 2. The last two sections deal with manifolds of non-positive curvature. Here, in particular the case of strictly negative curvature is treated for the first time in a monograph, with elementary proofs of many of the basic results in this important area.

After this brief description of the contents, it would certainly take more space to describe what has been omitted. To have an idea of topics not covered in this book see e.g. Chern [2] and de Rham [2]. Most notable is the absence of integration methods. It is clear that on Hilbert manifolds differential forms are bound to play a lesser rôle than on manifolds of finite dimension. But the deeper explanation for this and for most of the other omissions is simply that a book on mathematics, like any other literary work, is necessarily prejudiced by the personal experiences of the author and thus reveals strong autobiographical traits. As a matter of fact, I have composed this monograph around my own area of research in Riemannian geometry over the past 25 years, thereby including the work of younger colleagues who I had the privilege of meeting to mutual advantage. I have no other excuse to proffer for the selection of the contents, except that I am convinced that my choice represents a lasting contribution to the field and that future fruitful developments seem most likely.

Thus, I hope that my efforts in writing this book over many years will not just be a record of results and methods but will also serve as an impetus towards further research.

It only remains to express my gratitude to the people who helped me with the manuscript, by reading whole sections. I wish to mention in particular W. Ballmann, V. Bangert, J. Eschenburg, H. Matthias, A. Thimm, G. Thorbergsson and F. Wolter. Finally, I wish to thank Walter de Gruyter & Co. for accepting my manuscript in their new series 'Studies in Mathematics'.

Bonn, 1982

Wilhelm Klingenberg

Table of Contents

Chapter 1: Foundations

This chapter contains the basic definitions and results on differentiable manifolds, vector and tensor bundles over such manifolds and Riemannian metrics. The material presented here differs little from that in other well-known text books, except that we consider manifolds modelled on Hilbert spaces rather than on finite dimensional spaces. This will be useful in Chapter 2 and presents no conceptual difficulties anyway, as was demonstrated by Lang [1].

Not quite standard in our chapter on the Foundations is the discussion of submersions (see 1.11) and Jacobi fields (see 1.12). This constitutes a first step towards global geometry, which is the subject of the remainder of the book.

1.0 Review of Differential Calculus and Topology

In this section we set forth some notation and recall some basic properties of differentiable maps between Banach spaces. For details we refer to Dieudonné [1] and Lang [1]. We shall conclude with some facts on topological spaces. Reference will be made to Bourbaki [1].

1. We denote by $\mathbb{E}, \mathbb{E}', \mathbb{E}_i, \ldots, \mathbb{F}, \mathbb{F}', \mathbb{F}_i, \ldots$ real Banach spaces. In fact, most of the time these will actually be separable (complete) Hilbert spaces. Subspaces are always assumed to be closed and linear mappings are assumed to be continuous.

Note. Subspaces of finite dimension or finite codimension are always closed.

We say that a closed (linear) subspace $\mathbb{E}'$ of $\mathbb{E}$ *splits* if there exists a closed complement $\mathbb{E}''$ such that $\mathbb{E}$ is isomorphic to $\mathbb{E}' \times \mathbb{E}''$. Note that for a Hilbert space $\mathbb{E}$, every subspace $\mathbb{E}'$ splits: Take for $\mathbb{E}''$ the orthogonal complement of $\mathbb{E}'$.

Let $F\colon \mathbb{E} \to \mathbb{F}$ be an injective linear mapping whose image is a closed subspace $\mathbb{F}'$. F is called a *splitting* mapping if $\mathbb{F}'$ splits, i.e., if $\mathbb{F} \cong \mathbb{F}' \times \mathbb{F}''$. More generally, a linear mapping $F\colon \mathbb{E} \to \mathbb{F}$ with closed image is called a splitting mapping if the induced injection $\mathbb{E}/\mathit{ker}\, F \to \mathbb{F}$ splits. Again, for Hilbert spaces, any closed linear mapping splits – closed means that the image is a closed subspace.

Let us denote by $L(\mathbb{E}; \mathbb{F})$ the vector space of linear mappings $F\colon \mathbb{E} \to \mathbb{F}$. $L(\mathbb{E}; \mathbb{F})$ becomes a Banach space by taking as norm $|F|$ of an $F \in L(\mathbb{E}; \mathbb{F})$ the greatest lower bound of all numbers k such that

$$|F.X| \leqslant k|X|, \quad \text{for all } X \in \mathbb{E}.$$

If $\mathbb{E}$ and $\mathbb{F}$ are finite dimensional, one can define a scalar product on $L(\mathbb{E}, \mathbb{F})$ so as to make it into a (finite dimensional) Hilbert space, see (1.0.2).

More generally, we define a norm on the space

$$L(\mathbb{F}_1, \ldots, \mathbb{F}_r; \mathbb{G})$$

of r-linear mappings from $\mathbb{F}_1 \times \ldots \times \mathbb{F}_r$ into $\mathbb{G}$ by taking for $|F|$ the greatest lower bound of real numbers k satisfying

$$|F(X_1, \ldots, X_r)| \leqslant k|X_1| \ldots |X_r|,$$

where $(X_1, \ldots, X_r) \in \mathbb{F}_1 \times \ldots \times \mathbb{F}_r$.

With this, the canonical mapping

$$L\big(\mathbb{F}_1; L(\mathbb{F}_2; \ldots; L(\mathbb{F}_r; \mathbb{G})\big) \to L(\mathbb{F}_1, \mathbb{F}_2, \ldots, \mathbb{F}_r; \mathbb{G})$$

from the space of iterated linear maps into the space of multilinear maps becomes a Banach space isomorphism.

Of particular importance are the various tensor spaces associated to a Hilbert space $\mathbb{E}$. Let $\mathbb{E}^*$ denote the dual of $\mathbb{E}$. Then

$$T_s^r\,\mathbb{E} \equiv L(\underbrace{\mathbb{E}^*, \ldots, \mathbb{E}^*}_{r}, \underbrace{\mathbb{E}, \ldots, \mathbb{E}}_{s}; \mathbb{R})$$

is called the *space of r-fold contravariant and s-fold covariant tensors*.

We also use $T_s^r\,\mathbb{E}$ to denote any of the $(r+s)!/r!s!$ spaces $L(\mathbb{E}_1, \ldots, \mathbb{E}_{r+s}; \mathbb{R})$, where r of the $\mathbb{E}_i$ are equal to $\mathbb{E}^*$ and the remaining s of the $\mathbb{E}_i$ are equal to $\mathbb{E}$.

Since $L(\mathbb{E}^*; \mathbb{R}) = \mathbb{E}$; $L(\mathbb{E}; \mathbb{R}) = \mathbb{E}^*$ we have for $rs > 0$

$$T_s^r\,\mathbb{E} = L(\underbrace{\mathbb{E}^*, \ldots, \mathbb{E}^*}_{r}, \underbrace{\mathbb{E}, \ldots, \mathbb{E}}_{s}; \mathbb{R}) \cong L(\underbrace{\mathbb{E}^*, \ldots, \mathbb{E}^*}_{r-1}, \underbrace{\mathbb{E}, \ldots, \mathbb{E}}_{s}; \mathbb{E})$$

$$\cong L(\underbrace{\mathbb{E}^*, \ldots, \mathbb{E}^*}_{r}, \underbrace{\mathbb{E}, \ldots, \mathbb{E}}_{s-1}; \mathbb{E}^*)$$

and

$$T_1^0\,\mathbb{E} \cong \mathbb{E}^*; \; T_0^1\,\mathbb{E} \cong \mathbb{E}.$$

A word to explain the terminology: Take e.g. $X \in T_0^1\,\mathbb{E} = \mathbb{E}$, a 1-fold contravariant tensor. Let $F\colon \mathbb{E} \to \mathbb{E}$ be an automorphism. Choose an (orthonormal) Hilbert basis $\{e_i\}$ and its dual $\{e^i\}$. Then the i-th coordinate of X ist given by $X^i = \langle e^i, X\rangle$ where we denote by $\langle\,,\,\rangle$ the *canonical pairing* $\mathbb{E}^* \times \mathbb{E} \to \mathbb{R}$. The i-th coordinate of FX is given by $X'^i = \langle e^i, FX\rangle = \langle {}^tFe^i, X\rangle$, ${}^tF\colon \mathbb{E}^* \to \mathbb{E}^*$ being the transpose of F. That is, $X'^i = \sum_k ({}^tF)^i_k X^k$ where the $({}^tF)^i_k$ are the elements of the transposed matrix ${}^tF$ of F. Thus, the coordinates of a vector are transformed with the transposed matrix which is why X is called contravariant.

Let $\mathbb{F}$ be a Banach space. By $GL(\mathbb{F})$ we mean the group of (linear) automorphisms of $\mathbb{F}$. Consider a tensor space $T_s^r\,\mathbb{E}$ of the Hilbert space $\mathbb{E}$. Then we have a canonical group morphism

$$T_s^r\colon GL(\mathbb{E}) \to GL(T_s^r\,\mathbb{E})$$

given by associating with $\{F\colon \mathbb{E} \to \mathbb{E}\} \in GL(\mathbb{E})$ the mapping

$$T_s^r F\colon T_s^r\,\mathbb{E} \to T_s^r\,\mathbb{E};$$

$$X_s^r \mapsto X_s^r \circ (\underbrace{{}^tF \times \ldots \times {}^tF}_{r} \times \underbrace{F^{-1} \times \ldots \times F^{-1}}_{s}),$$

where ${}^tF\colon \mathbb{E}^* \to \mathbb{E}^*$ is the transpose of F.

Indeed, one verifies at once that $T_s^r F$ is linear and that $T_s^r(F_2 \circ F_1) = T_s^r F_2 \circ T_s^r F_1$. Moreover, $T_s^r(id|\,\mathbb{E}) = id|\,\mathbb{E}_s^r$. Thus T_s^r is what is called in category theory *a covariant functor*.

A subspace of $T_s^r\,\mathbb{E}$ which is invariant under the subgroup $T_s^r GL(\mathbb{E})$ of $GL(T_s^r\,\mathbb{E})$ is called a *(general) tensor space*.
We give two important examples of such general tensor spaces in $T_s^0\,\mathbb{E} = L(\underbrace{\mathbb{E}, \ldots, \mathbb{E}}_{s}; \mathbb{R})$:

(i) The *space* $S_s\,\mathbb{E}$ *of s-fold covariant and symmetric tensors* consists of those elements $Z_s^0 \in T_s^0\,\mathbb{E}$ which satisfy

$$Z_s^0(X_{\sigma(1)}, \ldots, X_{\sigma(s)}) = Z_s^0(X_1, \ldots, X_s),$$

for all permutations σ of the set $\{1, \ldots, s\}$.

(ii) The *space* $A_s E$ *of s-fold covariant antisymmetric tensors* consisting of the $Z_s^0 \in T_s^0\,\mathbb{E}$ satisfying

$$Z_s^0(X_{\sigma(1)}, \ldots, X_{\sigma(s)}) = \operatorname{sign}\sigma\, Z_s^0(X_1, \ldots, X_s),$$

for all permutations σ of $\{1, \ldots, s\}$.

We see that if $\dim \mathbb{E} = n < \infty$, then

$$\dim S_s\,\mathbb{E} = (n+s-1)!/s!\,(n-1)! \quad \text{whereas}$$

$$\dim A_s\,\mathbb{E} = \begin{cases} n!/s!\,(n-s)!, & \text{for } 1 \leqslant s \leqslant n \\ 0 \quad , & \text{for } s > n \end{cases}$$

We conclude this section by indicating some canonical isomorphisms between spaces of linear maps *when all vector spaces have finite dimension.*

Recall that the *tensor product* $\mathbb{E} \otimes \mathbb{F}$ of two vector spaces $\mathbb{E}$ and $\mathbb{F}$ is characterized by the following properties:

(i) There exists a bilinear mapping

$$\Phi\colon \mathbb{E} \times \mathbb{F} \to \mathbb{E} \otimes \mathbb{F};\ (X, Y) \mapsto X \otimes Y$$

such that the image generates $\mathbb{E} \otimes \mathbb{F}$ as a vector space.

(ii) Given any bilinear map $F \in L(\mathbb{E}, \mathbb{F}; \mathbb{G})$, there exists a unique $G \in L(\mathbb{E} \otimes \mathbb{F}; \mathbb{G})$ with $F = G \circ \Phi$.

In particular, if $\{e_i\}$, $\{f_j\}$ are bases of $\mathbb{E}$ resp. $\mathbb{F}$, then $\{e_i \otimes f_j\}$ is a basis for $\mathbb{E} \otimes \mathbb{F}$.

One has the canonical isomorphisms:

$$L(\mathbb{E} \otimes \mathbb{F}; \mathbb{G}) \cong L(\mathbb{E}, \mathbb{F}; \mathbb{G}) \cong L\big(\mathbb{E}; L(\mathbb{F}; \mathbb{G})\big)$$

$$L(\mathbb{E}; \mathbb{F}^*) \cong \mathbb{E}^* \otimes \mathbb{F}^*$$

For instance, $F \in L(\mathbb{E}; \mathbb{F}^*)$ corresponds to $\sum_{i,j} \langle f_j, F(e_i)\rangle\, e^i \otimes f^j$ where $\{e^i\}, \{f^j\}$ are the dual bases of the bases $\{e_i\}$, $\{f_j\}$ of $\mathbb{E}$ and $\mathbb{F}$. Combining these isomorphisms, we get the

1.0.1 Proposition. *Let* $\dim \mathbb{E} < \infty$. *Then*

$$T_s^r\, \mathbb{E} \cong \underbrace{\mathbb{E} \otimes \ldots \otimes \mathbb{E}}_{r} \otimes \underbrace{\mathbb{E}^* \otimes \ldots \otimes \mathbb{E}^*}_{s}$$

and also

$$T_s^r\, \mathbb{E} \cong L(\underbrace{\mathbb{E}, \ldots, \mathbb{E}}_{s}; \underbrace{\mathbb{E} \otimes \ldots \otimes \mathbb{E}}_{r}) =$$

$$L(\underbrace{\mathbb{E}, \ldots, \mathbb{E}}_{s-1}; \mathbb{E}^* \otimes \underbrace{\mathbb{E} \otimes \ldots \otimes \mathbb{E}}_{r}) \quad etc. \quad \square$$

Note. The concept of the projective tensor product for Banach spaces allows one to extend these isomorphisms to the case of infinite dimensions. See Schatten [1].

For later use we point out another feature of vector spaces $\mathbb{E}$ of finite dimension.

1.0.2 Proposition. *Let* $\dim \mathbb{E} = n < \infty$. *Let* $\langle , \rangle$ *denote the scalar product on* $\mathbb{E}$. *Then on* $T_s^r\, \mathbb{E}$ *this determines intrinsically a scalar product as follows: Let* $\{e_i\}$, $1 \leqslant i \leqslant n$, *be an orthonormal basis for* $\mathbb{E}$. *Together with the dual basis* $\{e^j\}$, $i \leqslant j \leqslant n$, *the*

$$e_{i_1 \ldots i_r}^{j_1 \ldots j_s} = e_{i_1} \otimes \ldots \otimes e_{i_r} \otimes e^{j_1} \otimes \ldots \otimes e^{j_s}$$

$$for\ \ 1 \leqslant i_i, \ldots, i_r \leqslant n;\ 1 \leqslant j_1, \ldots, j_s \leqslant n$$

form a basis for $T_s^r\, \mathbb{E}$. *Now define the metric on* $T_s^r\, \mathbb{E}$ *by letting this basis be orthonormal.*

This definition is independent of the choice of the orthonormal bases $\{e_i\}$.

Proof. Any two orthonormal bases $\{e_i\}$, $\{e_i'\}$ are related by an orthogonal transformation $A = (a_i^k)$:

$$e_i' = \sum_k a_i^k e_k, \ \sum_k a_i^k a_j^k = \delta_{ij}.$$

The corresponding bases

$$\{e_{i_1 \ldots i_r}^{j_1 \ldots j_s}\}; \quad \{e'^{j_1 \ldots j_s}_{i_1 \ldots i_r}\}$$

then are related by the linear transformation $T_s^r A$, i.e.,

$$e'^{j_1 \dots j_s}_{i_1 \dots i_r} = \sum_{\substack{k_1, \dots, k_r \\ l_1, \dots, l_s}} a^{k_1}_{i_1} \dots a^{k_r}_{i_r} b^{j_1}_{l_1} \dots b^{j_s}_{l_s} e^{l_1 \dots l_s}_{k_1 \dots k_r}$$

Here $e'^j = \sum_l b^j_l e^l$, i.e., (b^j_l) ist the transposed inverse or contragradient ${}^tA^{-1}$ of A.

From $A^tA = E$, $({}^tA^{-1})^t({}^tA^{-1}) = E$ we see that the $e'^{j_1 \dots j_s}_{i_1 \dots i_r}$ also form an orthonormal basis. That is to say, $A \in \mathbb{O}(\mathbb{E})$ implies $T_s^r A \in \mathbb{O}(T_s^r \mathbb{E})$. □

2. Let $U \subset \mathbb{F}$, $U' \subset \mathbb{F}'$ be open sets. Let $F\colon U \to U'$ be a mapping. F is called *differentiable at* $u_0 \in U$ if there exists a $DF(u_0) \in L(\mathbb{F}; \mathbb{F}')$ such that

$$F(u) - F(u_0) - DF(u_0) \,.\, (u - u_0) = o(|u - u_0|)\,.$$

Here, $o(r)$ satisfies $\lim_{r \to 0; r > 0} \|o(r)/r\| = 0$. F is called *differentiable of class* C^1 if it is differentiable for all $u \in U$ and $u \in U \mapsto DF(u) \in L(\mathbb{F}; \mathbb{F}')$ is continuous.

That $F\colon U \to U'$ is *differentiable of class* C^r is defined by induction. Assume we have defined $D^{r-1}F$ as a mapping from U into $L(\mathbb{F}; L(\mathbb{F}; \dots L(\mathbb{F}; \mathbb{F}')))$ which we can identify with $L(\mathbb{F}, \mathbb{F}, \dots; \mathbb{F}')$, with $(r-1)$ times $\mathbb{F}$. If $D^{r-1}F$ is differentiable of class C^1, put $D(D^{r-1}F) = D^rF$ and call F differentiable of class C^r.

Finally, we call $F\colon U \to U'$ *differentiable*, if it is differentiable of class C^r for all r.

Sometimes we will find it convenient to use the language of categories and functors. Thus we may speak of the category formed by the open subsets of Banach spaces as objects and the differentiable mappings between them as morphisms. This means that with

$$F_1\colon U_1 \subset \mathbb{E}_1 \to U_2 \subset \mathbb{E}_2;\ F_2\colon U_2 \subset \mathbb{E}_2 \to U_3 \subset \mathbb{E}_3,$$

being differentiable the composition

$$F_2 \circ F_1\colon U_1 \to U_3$$

is also differentiable. Moreover, $id_U\colon U \subset \mathbb{E} \to U \subset \mathbb{E}$ is differentiable.

Let $U \subset \mathbb{E}$ be open. For every $u_0 \in U$ we define the *tangent space* $T_{u_0}U$ *of* U *at* u_0 as the set $\{(u_0, X); X \in \mathbb{E}\}$, endowed with the vector space structure arising from the canonical mapping

$$pr_2\colon (u_0, X) \in T_{u_0}U \mapsto X \in \mathbb{E}$$

The collection of the tangent spaces $T_{u_0}U$, $u_0 \in U$, is denoted by TU. The canonical isomorphism

$$TU \cong U \times \mathbb{E}$$

makes TU into an open subset of $\mathbb{E} \times \mathbb{E}$.

The projection $pr_1\colon U \times \mathbb{E} \to U$ onto the first factor will also be written as

$$\tau \equiv \tau_U\colon TU \to U;\ (u_0, X) \mapsto u_0\,.$$

τ_U is called *tangent bundle of* U. TU is called the *total tangent space of* U and τ ist called the *projection* of the tangent bundle.

For a differentiable

$$F\colon U \subset \mathbb{E} \to V \subset \mathbb{F}$$

we define the *tangential of F*,

$$TF\colon TU \to TV,$$

by $(u, X) \mapsto (F(u), DF(u) . X)$.

Note that, for each $u \in U$, the restriction $T_u F = TF | T_u U$ is a linear mapping which is completely determined by the differential $DF(u)\colon \mathbb{E} \to \mathbb{F}$. For this reason, $DF(u)$ and $T_u F$ sometimes are identified. But basically, $DF(u)$ is a mapping from $\mathbb{E}$ to $\mathbb{F}$ while $T_u F$ is a mapping between tangent spaces of $\mathbb{E}$ and $\mathbb{F}$.

Associating with $F\colon U \to V$ its tangential $TF\colon U \times \mathbb{E} \to V \times \mathbb{F}$ constitutes a covariant functor from the category of differentiable mappings between open sets into the same category. Indeed, let

$$F_1\colon U_1 \subset \mathbb{E}_1 \to U_2 \subset \mathbb{E}_2;\ F_2\colon U_2 \subset \mathbb{E}_2 \to U_3 \subset \mathbb{E}_3$$

be morphisms. Then the morphisms

$$T(F_2 \circ F_1)\colon T_1 U_1 \to TU_3;\ TF_2 \circ TF_1\colon TU_1 \to TU_3$$

are the same. And the tangential Tid_U of the identity mapping $id_U\colon U \to U$ is the identity mapping $id_{TU}\colon TU \to TU$.

Actually, the tangential is a special sort of morphism; it preserves the product structure $U \times \mathbb{E}$ of the objects TU. This amounts to the commutativity of the diagram

$$\begin{array}{ccc} U \times \mathbb{E} = TU & \xrightarrow{\quad TF \quad} & TV = V \times \mathbb{E} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\quad F \quad} & V \end{array}$$

Moreover, the restrictions $T_u F = TF | T_u U$ are linear. Therefore we may say that the pair (F, TF) becomes a morphism in the category of tangent bundles of the open sets U of Banach spaces.

3. We continue with the inverse mapping theorem and two corollaries concerning locally injective and locally surjective differentiable mappings. For our later applications it suffices to consider the case that all spaces are Hilbert spaces.

1.0.3 Theorem. *Let U be an open neighborhood of $0 \in \mathbb{E}$. Let*

(*) $\qquad F\colon U \to \mathbb{F},\ F(0) = 0$

be differentiable such that $DF(0)\colon \mathbb{E} \to \mathbb{F}$ is a (bijective) isomorphism. Then F is a local *diffeomorphism. That is to say, there exist open neighborhoods* U', V' of $0 \in \mathbb{E}$, $0 \in \mathbb{F}$, $U' \subset U$, *such that* $F|U'\colon U' \to V'$ *is a diffeomorphism.*

A *diffeomorphism* is a differentiable homeomorphism such that also the inverse is differentiable.

1.0.4 Corollary 1. *Assume that the mapping* F, (*), *has the property that* $DF(0)\colon \mathbb{E} \to \mathbb{F}$ *is an isomorphism with a closed subspace* $\mathbb{F}_1$ *of* $\mathbb{F}$. *Write* $\mathbb{F} = \mathbb{F}_1 \times \mathbb{F}_2$. *Then there exists a local diffeomorphism*

$$g\colon \mathbb{F} \to \mathbb{F}_1 \times \mathbb{F}_2;\ g(0) = 0$$

and an open neighborhood $U_1 \subset U$ *of* $0 \in \mathbb{E}$ *such that*

$$g \circ (F|U_1)\colon U_1 \to U_1 \times \{0\} \subset \mathbb{E} \times \{0\} \cong \mathbb{F}_1 \times \{0\} \subset \mathbb{F}$$

is the canonical linear injection.

1.0.5 Corollary 2. *Assume that the mapping* F, (*), *has the property that* $DF(0)\colon \mathbb{E} \to \mathbb{F}$ *is surjective. Write* $\mathbb{E} = \mathbb{E}_1 \times \mathbb{E}_2$ *with* $DF(0)|\mathbb{E}_2\colon \mathbb{E}_2 \to \mathbb{F}$ *bijective, i.e.,* $\mathbb{E}_2 \cong \mathbb{F}$ *via* $DF(0)|\mathbb{E}_2$. *Then there exists a local diffeomorphism*

$$h\colon (U_1 \times U_2, 0) \subset (\mathbb{E}_1 \times \mathbb{E}_2, 0) \to (\mathbb{E}, 0)$$

with U_i *an open neighborhood of* $0 \in \mathbb{E}_i$ *such that*

$$F \circ h\colon U_1 \times U_2 \to U_2 \subset \mathbb{E}_2 \cong \mathbb{F}$$

is the projection pr_2 *onto the second factor.*

Note. If $\mathbb{E}$, $\mathbb{F}$ are Banach spaces, one must assume that ker $DF(0)$ splits.

Proof. For the proof of (1.0.3) one uses the contraction lemma. For details we refer to the literature, cf. Dieudonné [1], Lang [1].

Corollary 1 is deduced from the theorem by extending $F\colon U \to \mathbb{F}$ to a locally invertible mapping

$$\Phi\colon U \times \mathbb{F}_2 \subset \mathbb{E} \times \mathbb{F}_2 \cong \mathbb{F} \to \mathbb{F}_1 \times \mathbb{F}_2;\quad (u, v_2) \mapsto F(u) + (0, v_2).$$

Indeed, $D\Phi(0, 0) = DF(0) + (0, id|\mathbb{F}_2)$. Taking g as the local inverse of Φ we prove our claim.

Similarly, for the proof of Corollary 2, we consider

$$\Phi\colon U \subset \mathbb{E}_1 \times \mathbb{E}_2 \cong \mathbb{E}_1 \times \mathbb{F} \to \mathbb{E}_1 \times \mathbb{E}_2;\quad (u_1, u_2) \mapsto (u_1, F(u_1, u_2)).$$

Then $D\Phi(0) = (id|\mathbb{E}_1, 0) + (DF(0)|\mathbb{E}_1, DF(0)|\mathbb{E}_2)$, i.e., $D\Phi(0)$ is bijective. Taking for h the local inverse of Φ we get a mapping satisfying our requirements.

4. A topological space M is called *metrizable* if there exists a metric on M which induces the given topology.

M is called *separable* if it possesses a countable base for the open sets. For metric spaces this is equivalent to saying that there exists a countable dense set of points in M.

Let $(M_\alpha)_{\alpha\in A}$ be an open covering of M. That is, all M_α are open and every $p \in M$ is contained in some M_α. An open covering $(\tilde{M}_\beta)_{\beta\in B}$ of M will be called *refinement* of the covering $(M_\alpha)_{\alpha\in A}$ if there exists a mapping $\sigma: B \to A$ such that $\tilde{M}_\beta \subset M_{\sigma(\beta)}$.

An open covering $(M_\alpha)_{\alpha\in A}$ of M is called *locally finite* if every point $p \in M$ possesses a neighborhood U such that $U \cap M_\alpha \neq \emptyset$ for finitely many α only.

A topological space M is called *paracompact* if every open covering of M possesses a locally finite refinement. Clearly, a compact space is paracompact. So are finite dimensional Banach spaces. An important sufficient condition for a space to be paracompact is that it is metrizable.

The property of a topological space to be metrizable is preserved under the operations of forming products and taking subsets. Our interest in this property stems from the fact that we will be considering Riemannian manifolds on which the Riemannian structure defines a metric which induces the given topology.

A *partition of unity* on a topological space M consists of a family $(\phi_\beta, \tilde{M}_\beta)_{\beta\in B}$. Here, $(\tilde{M}_\beta)_{\beta\in B}$ forms a locally finite open covering of M and $\phi_\beta: M \to \mathbb{R}$ is continuous $\geqslant 0$ with $\{\phi_\beta > 0\} \subset \tilde{M}_\beta$ and $\sum_\beta \phi_\beta(p) = 1$, for all $p \in M$.

A topological space M is said to *admit partitions of unity* if, for every open covering $(M_\alpha)_{\alpha\in A}$, there exists a partition of unity $(\phi_\beta, \tilde{M}_\beta)_{\beta\in B}$ with $(\tilde{M}_\beta)_{\beta\in B}$ being a refinement of $(M_\alpha)_{\alpha\in A}$. A paracompact separable space admits partitions of unity.

If M is a differentiable manifold in the sense of definition (1.1.2) then M even admits *differentiable partitions of unity*, i.e., the functions $\phi_\beta: M \to \mathbb{R}$ are of class C^∞. Cf. Lang [1]. For the finite dimensional case cf. also Hirsch [1] and Sulanke und Wintgen [1].

1.1 Differentiable Manifolds

In this section we introduce the concept of a differentiable manifold M, modelled on a (separable) Hilbert space $\mathbb{E}$. Essentially, this is a topological space which locally looks like an open set U of $\mathbb{E}$. Such a local representation of M is called a chart. So far, M is only a topological manifold. What makes M into a differentiable manifold is that the transition mappings, determined by the overlap of two charts, are diffeomorphisms, cf. (1.1.2).

The morphisms, i.e., the structure preserving mappings between differentiable manifolds, are introduced in (1.1.4). With this, we get the category of differentiable manifolds and mappings, see (1.1.5).

We conclude by showing that differentiable mappings can be localized. This means that every differentiable mapping, defined on some open set of a manifold M, when restricted to a suitable open neighborhood of a point $p \in M$, can be viewed as the restriction of a differentiable mapping defined on all of M. Thus, for local properties, there is no difference between local and global morphisms.