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FINANCIAL GEOMETRY

a geometric approach to hedging and risk management

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FINANCIAL TIMES

ALVIN KURUC

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This book is dedicated to the memory of my father Andrew William Kuruc, 1919–2002.
I wish he were still around to see it.

Foreword

Why might you choose to read this book? What was the author reaching to achieve in its writing? Some finance books offer a new perspective on the practitioner's view of a fashionable area of market activity: how clients can be served by a class of traded instruments, perhaps, or strategies for packing them, or outlines of techniques with which traders price and hedge. Other works offer a survey of various theoretical models, or a polemical defence of a new one. Just occasionally, a book comes along that actually helps the reader to work through the messy, specific mechanics of implementing these ideas as concrete calculations on real transactions.

This book, usually, offers none of these rewards for the time spent in its reading. The details of specific asset classes and their buy-side utility are not covered, nor are trading strategies discussed. The existence of valuation algorithms for every type of product is assumed but not described. Nor does the book come with a CD of spreadsheet add-ins or code samples that illustrate useful day-to-day techniques. Rather, this book provides a theoretical platform for the systematic analysis of mixed portfolios. The thematic question is: 'Given that the constituents of a portfolio are diverse, what can we say about the portfolio's overall behaviour?' Portfolio diversity arises primarily form the fact that portfolios can contain many different financial assets and liabilities, such as convertible bonds, listed equity derivatives and credit default swaps; or electicity forwards, options on aluminium futures and weather derivatives; or swaptions and caps and bond futures; or all of these and many more. A second dimension of heterogeneity arises when, as is usually the case, the various financial instruments are each valued using an approach appropriate to the conventions of its own market rather than under an 'ubermodel' covering all asset classes. Sometimes these models can even be directly inconsistent: for example, assuming that both LIBOR forwards and swap rates are lognormally distributed. A third form of variability arises from the fact that the valuation method appropriate even to an individual traded product is not set in stone and any methodology for protfolio analysis has to have the ability to accommodate alternative valuation techniques as a matter of course. For example, a more sophisticated approach to volatility may be adopted for a particular traded product at a particular time, and then only in some trading books.

Portfolio analysis is a broad term that covers any metric that can be applied to a collection of financial instruments. These forms of analysis that provide the most detailed insight naturally tend to rely more on the relatedness of the assets in the portfolio. An example could be a spot/vol matrix for a portfolio or equity of FX options. For a portfolio with a limited number of underlyings this can offer a rich source of information. On the other hand, when the number of underlyings increases into the dozens or higher such a matrix quickly becomes unmanageable. At the other extreme, there are measures such as value at risk (VaR) that are more robust in the sense that they can be applied validly and conveniently to very diverse portfolios; but such measures tend to be more limited in informational forecasting.

We have a sense, of course, that there should be a continuity of analytic tools that progresses from specific market sensitivity measures, calculated either analytically or by bumping inputs, through the addition of various statistical or other aggregation techniques to more compact measures like VaR. However, in real life it is too easy used by traders on individual books and reductive tools used by risk managers for regulatory and other firm-wide reporting.

The register upon which this loss of continuity takes place is software. It is hard to provide traders with excellent software, and the demands that command the highest attention are, rightly, those that arise from the specifics of particular products and local markets, and the need for flexible pricing and sensitivity analysis on particular books. It is also hard to provide risk managers with excellent software. Here, the demands that command the highest attention are, rightly, those that permit the greatest breadth of coverage and robustness of measure. It is very hard to address these two constituencies simultaneously and to provide both with excellent software at the same time. Hardly any software houses or internal IT departments (apart from smaller ones, whose reduced resources often paradoxically impose more focus leading to better systems) even attempt it. However, it is the best approach. P&L reporting, sensitivity analysis, pricing grids, scenario analysis, value at risk measures, delta VaR, P&L decomposition, VaR back-testing, variance minimization, hedging, exposure profiling and all the rest of the tools of the trade are different computational cuts from a small palette of concepts. The extent to which this is realised in software corresponds to the extent to which the various parties at a trading firm share the same vocabulary. Traders, controllers and risk managers all benefit from having compatible analysis. They do different functions and so need different tools, but the functions and the tools are overlapping and benefit from having the same underlying architecture.

The financial aspect of this common architecture is described in this book. Software designers who work in this field will recognise congruencies with (existing or potential) systems architectures including, inter alia: the obsessive detail around the specification of time and currency; the need for a precise and transformable definition of risk factors, subtly related to market prices; the apparatus for change of basis; analysis of Jacobians, Hessians and operations on tangent spaces; and the range of techniques for hedging the sensitivities that these represent.

The language in which these abstractions are articulated - differential geometry - will be less familiar and comfortable to many. For readers who want to progress more speedily through the book, I can recommend starting off by just reading the pictures. Like me, you are likely at some point to glance at the text alongside and skid to a halt, realising that you have to back-track several chapters to catch up with the commentary underneath. Ultimately, the mathematical notation is efficient. The apparatus represented primarily in the notation of subscripts and superscripts seems cumbersome, but anyone who has successfully implemented robust software in this domain will recognise that there is an unfortunate level of detail that cannot be avoided. As an illustration of the effectiveness of the financial framework, the author has used this approach ingeniously elsewhere to articulate the elusive commensurability of different measures of volatility, along the lines that are hinted at in sections of this book. (See 'Comparing Apples and Oranges - Vega Risk Revisited', *Risk Professional* 2(6):28–30, July August 2000.)

Students of financial mathematics will find in this book an unusually coherent treatment of the key themes of portfolio analysis. Most of us who work in this space will naturally be inclined to transcribe this back into specifications of systems, as the author himself does by day. We can read and refer to the book for reassurance that the purposes of the trader, the controller and the risk manager can be addressed together.

Ian Green
Managing Director
Credit Suisse First Boston

Preface

While economics is sometimes called the dismal science, there is at least one area where high-powered mathematical reasoning has found real-world application. Sophisticated tools from stochastic calculus are now routinely used for the valuation of financial derivatives. Since the publication of the seminal papers on the subject over 30 years ago, the mathematical foundations for the valuation of financial derivatives have become well established. There can be little doubt that the recent proliferation of books on stochastic calculus has been largely stimulated by interest in financial applications. Indeed, there are now many excellent books that develop stochastic calculus in the context of its central role in these problems.

Although research remains active on many points of technical detail, there is now a large degree of consensus on the general approach to the valuation of the commonly traded financial derivatives. However, another important task in the practical management of financial derivatives, hedging and risk management, is much less standardized. While a number of fundamental concepts and techniques have become well accepted, efficient hedging and risk management of financial portfolios remains a challenging problem.

At the heart of most hedging and risk-management techniques is the idea that the value of financial instruments can be described as a function of a set of so-called risk factors. Usually, though not invariably, the risk factors are directly or indirectly tied to the market prices of liquid instruments. In any case, the fundamental assumption is that one can express the value of a portfolio as a function of the risk factors. Under this assumption, hedging and risk management become, in essence, an exercise in understanding how the risk factors might change and thereby change the value of the portfolio and the potential hedging instruments available for purchase in the market. Commonly used risk factors include foreign exchange and interest rates, asset prices, and implied volatilities of options.

Apart from residual difficulties in the valuation of individual financial instruments, the reasons why hedging and risk-management problems continue to be difficult include the following:

1 The *number* of risk factors can be very large.

2 The value of a financial instrument is generally a *nonlinear* function of the risk factors.

3 There is no *canonical* definition of the risk factors. Risk factors are not always just directly observable market prices and different operators and systems may choose to

parameterize equivalent risk factors in different ways. For example, forward LIBOR rates and swap rates are two equivalent ways of parameterizing the term structure of interest rates. Cap traders naturally look at the world in terms of forward LIBOR rates, while swap traders naturally look at the world in terms of swap rates.

4 *Inconsistent valuation assumptions* are often made for different classes of instruments. This makes it difficult to reconcile risk information computed at the level of individual instruments into a coherent picture of the risk of the portfolio as a whole.

5 The allowable risk-factor values are often a *low-dimensional nonlinear subspace* embedded in a higher-dimensional space. For example, the possible foreign-exchange rates between cross-currency pairs are restricted by arbitrage relationships. Thus the exchange rates for the euro and the yen relative to the dollar effectively determine the exchange rate for the euro relative to the yen.

6 Certain important classes of risk factors are *not directly observed*, but must be inferred from market prices that also depend on other, more primary, risk factors. For example, so-called implied volatilities are inferred from option prices that also depend strongly on the value of the asset underlying the option. Typically, these secondary risk factors must be estimated using information that is less reliable than that available for the primary risk factors.

On a practical level, most major financial institutions depend on a large number of heterogeneous computer systems to manage various kinds of trades. While it would be very attractive to have a single unified system for managing the various types of trades, the specialization, complexity, and rapid evolution of modern financial instruments make this very difficult to attain. Among the more vexing problems caused by the use of heterogeneous trading systems is the integration of risk information into a coherent whole at the enterprise level. While most trading systems provide some form of risk analysis, the semantics of the information from these risk analyses is not well standardized. For example, trading systems for derivatives generally supply risk measures such as delta and vega. However, due to varying modelling assumptions and normalization conventions, one cannot blithely add together the deltas and vegas supplied by various trading systems in order to obtain summary measures of risk at the enterprise level.

The central theme of this book is that another area of mathematics, differential geometry, provides a concise and precise language for describing and analyzing multivariate hedging and risk-management problems, much as stochastic calculus does for valuation problems. It exhibits how the conceptual framework, computational machinery, and intuitive visual metaphors of differential geometry can be gainfully applied to a variety of financial problems. Indeed, almost all of the common operations used in risk management can be presented in a unified way using a few fundamental concepts. A number of what might otherwise appear as unrelated *ad hoc* calculations will be seen to be variations on a common theme. For example, a myriad of calculations that convert perturbations, sensitivities, and covariance matrices based on different risk-factor representations are subsumed by a general change-of-coordinates machinery. The analysis of the manner in which yield-curve interpolation and valuation-model calibration effect risk calculations will be exhibited as instances of general calculations relating the valuation functions and sensitivities when a market-state space of lesser extent is embedded in a market-state space of greater extent. The same analysis applies to procedures for changing the base currency of risk calculations and the risk-factor-mapping techniques that are at the heart of practical variance/covariance value-at-risk implementations.

While the approach taken in this book is unconventional, the topics covered are, for the most part, bread-and-butter risk management. In particular, we take as a given the conventional 'risk-neutral' approach to valuation. A second or more advanced volume could well be written that takes a more holistic view of valuation and risk management. Moreover, much of the book is concerned with analyses based on first-order sensitivities. This should not be construed as suggesting that this most basic set of analyses will suffice for practical purposes. Rather, there is just a fair amount to say about first-order sensitivities and the lessons learned from them carry over to scenario analysis and more sophisticated forms of value-at-risk analysis. For example, most financial institutions compute value at risk using historical simulation rather than the simpler variance/covariance techniques presented in the book. However, the points made about the definition of risk factors and the generation of arbitrage-free perturbations of those risk factors apply to historical simulation as well as to the simpler methods that are presented here.

In order that the reader may have a sense of the big picture, there now follows a brief overview of the organization of the book as a whole.

The Prologue draws attention to some problems of units that are peculiar to financial calculations, drawing some analogies to the usage of units in physical calculations. Confusion due to the use of multiple systems of units is a common bane of physics and finance. However, finance has the additional burden that the ratios of the units commonly used for describing financial quantities vary over time.

Part I, Foundations, presents the technical foundation of the approach to risk taken in the book. It formalizes the notion of risk factors as coordinate systems on a market-state space. Sensitivities, or 'Greeks', are then presented as differentials of valuation functions on the market-state space. Particular attention is paid to reconciling the results obtained using different risk-factor coordinates on the same market-state space.

Part II, Asset values, extends the results obtained in Part I focusing on market-state spaces that consist essentially of asset values. While this includes stock and commodity prices, particular attention is paid to interest rates and foreign-exchange rates as risk factors. It develops the technical tools of embedding and projection maps to relate the valuation functions and sensitivities obtained when formalizing a given risk problem using market-state spaces of lesser or greater extent in the context of the interpolation of interest-rate curves. It also develops delta-hedging methodologies for hedging against changes in interest rates. The treatment of foreign-exchange rates develops a geometric picture of relative asset values as lines through the origin of Euclidean space and uses embedding and projection maps to facilitate change-of-base-currency calculations.

Part III, Metrics, introduces probabilistic models for changes in the market state. It develops value-at-risk, risk contributions, and hedging techniques based on multivariate Gaussian models. These techniques are given geometric interpretations in terms of the lengths and angles of risk-factor exposure vectors.

Part IV, Implied volatility, treats the implied volatility of options as a risk factor. It develops technical tools for computing sensitivities to curve-valued and surface-valued risk factors. A key goal of this part is the reconciliation of sensitivities obtained by operators who use inconsistent valuation models by relating them through the common market observables that are used to calibrate them.

The body of the book ends with a brief Epilogue. There are three technical appendices. Appendix A, The dimension of time, provides a more detailed treatment of time effects in financial risk management. Appendix B, Black–Scholes and Bachelier calculations, derives the valuation formulas and sensitivity calculations for the Black–Scholes and Bachelier models that are needed in the text. Appendix C, Index of notation, provides an index for the mathematical notations used in the book.

While I have endeavoured to explain all the financial concepts, it will help if the reader has some basic knowledge of financial markets and derivatives gained either through experience or from a basic textbook such as [Hul00]. In addition, it is assumed that the reader is comfortable with at least univariate differential and integral calculus and basic linear algebra. It is perhaps gratuitous to cite sources for these subjects, but we can recommend [CJ89a] for calculus and [Str88] for linear algebra. Finally, I have assumed some familiarity with the most basic notions of probability and statistics, such as a probability distribution. A good source for this is [PP01]. I have relegated a few comments that require a bit more background to the Technical Notes that are interspersed throughout the book.

I'd be delighted to hear corrections, comments, and suggestions from readers. I can be reached at alvin_kuruc@yahoo.com.

Alvin Kuruc
Claygate, Surrey
1 July 2003

Acknowledgements

This book is an outgrowth of my experiences in building financial-risk systems over the past six years. It attempts to summarize many of the key principles and ideas that I have found to be useful in thinking about these systems into a coherent whole.

I am greatly indebted to many of my former and present colleagues for what they have taught me and for stimulating my thoughts on the topics covered in the book. My long-time friend Jim Lewis introduced me to the field and much of the material in the book, notably that on asset-value sensitivities and hedging, had its origins in the work that we did together at Renaissance Software. My past and present colleague Ian Green has greatly influenced my thinking on the design of financial risk systems and stimulated much of the material on reconciling sensitivities and the modelling of sensitivities to volatilities. Other aspects of the book have been stimulated by my work with Jesse Wu, Bing Chin, Liang-Khoon Koh, Dima Pozdynakov, and Serguei/Isakov.

I thank Liang-Khoon Koh, Dean Tallam, Dima Pozdynakov, Ian Green, and Serguei Isakov for their feedback on drafts of the manuscript that were much harder to get through than the current one. Their feedback influenced the book probably more than they suspect.

The idea for the book was first expressed in words in the Spring of 2001 in a conversation with Craig Bouchard of NumeriX. It lay dormant for a while as I was fully engaged in the design and implementation of the NumeriX risk module. It came to life again following a fateful trip to Geneva to attend the ICBI Risk Management conference in December, 2001. At the conference, Nick Haining of NumeriX introduced me to Karen Davies of CIFT, to whom I described the idea for the book. Karen passed this on to Laurie Donaldson of Prentice Hall Financial Times, which soon led to the commissioning of the manuscript. On the way back to California, I stopped off in London and had a conversation with Ian Green, which soon led to me taking a job at Credit Suisse First Boston. I'd like to express my appreciation to Ian for his support for continuing on with the book project.

On a personal note, I'd like to express my appreciation to my friends and family for not disowning me after nine months of doing very little other than working. Special thanks are due to Jan Winn, Rose Winn, and Nap Morneau for looking after my daughter and cats during the move to London and Klaus Reinisch for including my daughter in his family outings while I was busy working. I thank my daughter Elizabeth for keeping the numerous versions of the draft manuscript updated with her delightful illustrations. Most of all, I thank my wife Donna for sticking with me despite having, all too often, to fend for herself in a new country.

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