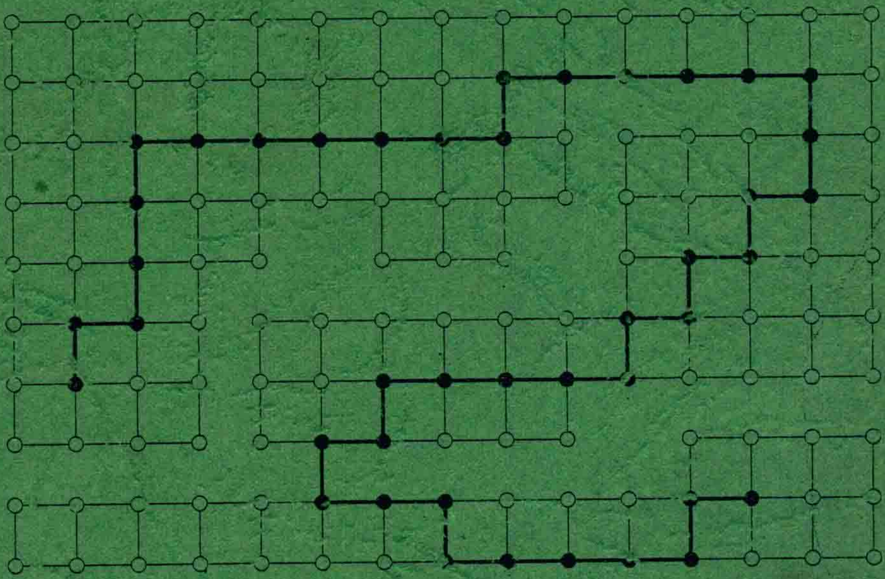
L. Lovász J. Pelikán K. Vesztergombi

DISCRETE MATHEMATICS

Elementary and Beyond



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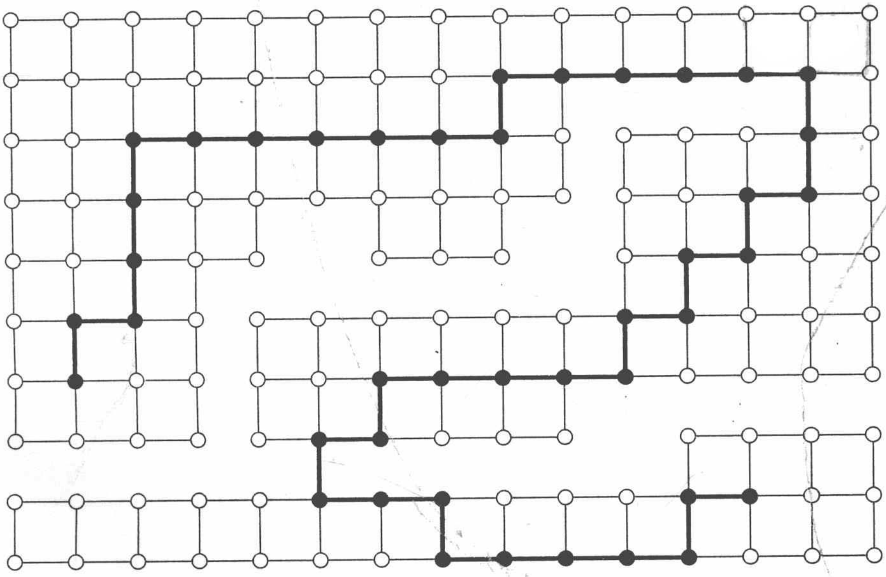

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Preface

For most students, the first and often only course in college mathematics is calculus. It is true that calculus is the single most important field of mathematics, whose emergence in the seventeenth century signaled the birth of modern mathematics and was the key to the successful applications of mathematics in the sciences and engineering.

But calculus (or analysis) is also very technical. It takes a lot of work even to introduce its fundamental notions like continuity and the derivative (after all, it took two centuries just to develop the proper definition of these notions). To get a feeling for the power of its methods, say by describing one of its important applications in detail, takes years of study.

If you want to become a mathematician, computer scientist, or engineer, this investment is necessary. But if your goal is to develop a feeling for what mathematics is all about, where mathematical methods can be helpful, and what kinds of questions do mathematicians work on, you may want to look for the answer in some other fields of mathematics.

There are many success stories of applied mathematics outside calculus. A recent hot topic is mathematical cryptography, which is based on number theory (the study of the positive integers $1, 2, 3, \ldots$), and is widely applied, for example, in computer security and electronic banking. Other important areas in applied mathematics are linear programming, coding theory, and the theory of computing. The mathematical content in these applications is collectively called *discrete mathematics.* (The word "discrete" is used in the sense of "separated from each other," the opposite of "continuous;" it is also often used in the more restrictive sense of "finite." The more everyday version of this word, meaning "circumspect," is spelled "discreet.")

The aim of this book is not to cover "discrete mathematics" in depth (it should be clear from the description above that such a task would be ill-defined and impossible anyway). Rather, we discuss a number of selected results and methods, mostly from the areas of combinatorics and graph theory, with a little elementary number theory, probability, and combinatorial geometry.

It is important to realize that there is no mathematics without *proofs*. Merely stating the facts, without saying something about why these facts are valid, would be terribly far from the spirit of mathematics and would make it impossible to give any idea about how it works. Thus, wherever possible, we will give the proofs of the theorems we state. Sometimes this is not possible; quite simple, elementary facts can be extremely difficult to prove, and some such proofs may take advanced courses to go through. In these cases, we will at least state that the proof is highly technical and goes beyond the scope of this book.

Another important ingredient of mathematics is *problem solving*. You won't be able to learn any mathematics without dirtying your hands and trying out the ideas you learn about in the solution of problems. To some, this may sound frightening, but in fact, most people pursue this type of activity almost every day: Everybody who plays a game of chess or solves a puzzle is solving discrete mathematical problems. The reader is strongly advised to answer the questions posed in the text and to go through the problems at the end of each chapter of this book. Treat it as puzzle solving, and if you find that some idea that you came up with in the solution plays some role later, be satisfied that you are beginning to get the essence of how mathematics develops.

We hope that we can illustrate that mathematics is a building, where results are built on earlier results, often going back to the great Greek mathematicians; that mathematics is alive, with more new ideas and more pressing unsolved problems than ever; and that mathematics is also an art, where the beauty of ideas and methods is as important as their difficulty or applicability.

László Lovász József Pelikán Katalin Vesztergombi

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Let's Count!

1.1 A Party

Alice invites six guests to her birthday party: Bob, Carl, Diane, Eve, Frank, and George. When they arrive, they shake hands with each other (strange European custom). This group is strange anyway, because one of them asks, "How many handshakes does this mean?"

"I shook 6 hands altogether," says Bob, "and I guess, so did everybody else."

"Since there are seven of us, this should mean $7 \cdot 6 = 42$ handshakes," ventures Carl.

"This seems too many" says Diane. "The same logic gives 2 handshakes if two persons meet, which is clearly wrong."

"This is exactly the point: Every handshake was counted twice. We have to divide 42 by 2 to get the right number: 21," with which Eve settles the issue.

When they go to the table, they have a difference of opinion about who should sit where. To resolve this issue, Alice suggests, "Let's change the seating every half hour, until we get every seating."

"But you stay at the head of the table," says George, "since it is your birthday."

How long is this party going to last? How many different seatings are there (with Alice's place fixed)?

Let us fill the seats one by one, starting with the chair on Alice's right. Here we can put any of the 6 guests. Now look at the second chair. If Bob

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sits in the first chair, we can put any of the remaining 5 guests in the second chair; if Carl sits in the first chair, we again have 5 choices for the second chair, etc. Each of the six choices for the first chair gives us five choices for the second chair, so the number of ways to fill the first two chairs is $5+5+5+5+5+5 = 6 \cdot 5 = 30$. Similarly, no matter how we fill the first two chairs, we have 4 choices for the third chair, which gives $6 \cdot 5 \cdot 4$ ways to fill the first three chairs. Proceeding similarly, we find that the number of ways to seat the guests is $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.

If they change seats every half hour, it will take 360 hours, that is, 15 days, to go through all the seating arrangements. Quite a party, at least as far as the duration goes!

1.1.1 How many ways can these people be seated at the table if Alice, too, can sit anywhere?

After the cake, the crowd wants to dance (boys with girls, remember, this is a conservative European party). How many possible pairs can be formed?

OK, this is easy: there are 3 girls, and each can choose one of 4 boys, this makes $3 \cdot 4 = 12$ possible pairs.

After ten days have passed, our friends really need some new ideas to keep the party going. Frank has one: "Let's pool our resources and win the lottery! All we have to do is to buy enough tickets so that no matter what they draw, we will have a ticket with the winning numbers. How many tickets do we need for this?"

(In the lottery they are talking about, 5 numbers are selected out of 90.)

"This is like the seating," says George. "Suppose we fill out the tickets so that Alice marks a number, then she passes the ticket to Bob, who marks a number and passes it to Carl, and so on. Alice has 90 choices, and no matter what she chooses, Bob has 89 choices, so there are $90 \cdot 89$ choices for the first two numbers, and going on similarly, we get $90 \cdot 89 \cdot 88 \cdot 87 \cdot 86$ possible choices for the five numbers."

"Actually, I think this is more like the handshake question," says Alice. "If we fill out the tickets the way you suggested, we get the same ticket more then once. For example, there will be a ticket where I mark 7 and Bob marks 23, and another one where I mark 23 and Bob marks 7."

Carl jumps up: "Well, let's imagine a ticket, say, with numbers $7, 23, 31, 34$, and 55. How many ways do we get it? Alice could have marked any of them; no matter which one it was that she marked, Bob could have marked any of the remaining four. Now this is really like the seating problem. We get every ticket $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ times."

"So," concludes Diane, "if we fill out the tickets the way George proposed, then among the $90 \cdot 89 \cdot 88 \cdot 87 \cdot 86$ tickets we get, every 5-tuple occurs not only once, but $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ times. So the number of *different* tickets is only

$$\frac{90 \cdot 89 \cdot 88 \cdot 87 \cdot 86}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}.$$

We only need to buy this number of tickets."

Somebody with a good pocket calculator computed this value in a twinkling; it was 43,949,268. So they had to decide (remember, this happens in a poor European country) that they didn't have enough money to buy so many tickets. (Besides, they would win much less. And to fill out so many tickets would spoil the party!)

So they decide to play cards instead. Alice, Bob, Carl and Diane play bridge. Looking at his cards, Carl says, "I think I had the same hand last time."

"That is very unlikely" says Diane.

How unlikely is it? In other words, how many different hands can you have in bridge? (The deck has 52 cards, each player gets 13.) We hope you have noticed that this is essentially the same question as the lottery problem. Imagine that Carl picks up his cards one by one. The first card can be any one of the 52 cards; whatever he picked up first, there are 51 possibilities for the second card, so there are $52 \cdot 51$ possibilities for the first two cards. Arguing similarly, we see that there are $52 \cdot 51 \cdot 50 \cdots 40$ possibilities for the 13 cards.

But now every hand has been counted many times. In fact, if Eve comes to kibitz and looks into Carl's cards after he has arranged them and tries to guess (we don't know why) the order in which he picked them up, she could think, "He could have picked up any of the 13 cards first; he could have picked up any of the remaining 12 cards second; any of the remaining 11 cards third.... Aha, this is again like the seating: There are $13 \cdot 12 \cdots 2 \cdot 1$ orders in which he could have picked up his cards."

But this means that the number of *different* hands in bridge is

$$\frac{52 \cdot 51 \cdot 50 \cdots 40}{13 \cdot 12 \cdots 2 \cdot 1} = 635{,}013{,}559{,}600.$$

So the chance that Carl had the same hand twice in a row is one in 635,013,559,600, which is very small indeed.

Finally, the six guests decide to play chess. Alice, who just wants to watch, sets up three boards.

"How many ways can you guys be matched with each other?" she wonders. "This is clearly the same problem as seating you on six chairs; it does not matter whether the chairs are around the dinner table or at the three boards. So the answer is 720 as before."

"I think you should not count it as a different pairing if two people at the same board switch places," says Bob, "and it shouldn't matter which pair sits at which board."

"Yes, I think we have to agree on what the question really means," adds Carl. "If we include in it who plays white on each board, then if a pair switches places we do get a different matching. But Bob is right that it doesn't matter which pair uses which board."

"What do you mean it does not matter? You sit at the first board, which is closest to the peanuts, and I sit at the last, which is farthest," says Diane.

"Let's just stick to Bob's version of the question" suggests Eve. "It is not hard, actually. It is like with handshakes: Alice's figure of 720 counts every pairing several times. We could rearrange the 3 boards in 6 different ways, without changing the pairing."

"And each pair may or may not switch sides" adds Frank. "This means $2 \cdot 2 \cdot 2 = 8$ ways to rearrange people without changing the pairing. So in fact, there are $6 \cdot 8 = 48$ ways to sit that all mean the same pairing. The 720 seatings come in groups of 48, and so the number of matchings is $720/48 = 15$."

"I think there is another way to get this," says Alice after a little time. "Bob is youngest, so let him choose a partner first. He can choose his partner in 5 ways. Whoever is youngest among the rest can choose his or her partner in 3 ways, and this settles the pairing. So the number of pairings is $5 \cdot 3 = 15$."

"Well, it is nice to see that we arrived at the same figure by two really different arguments. At the least, it is reassuring" says Bob, and on this happy note we leave the party.

1.1.2 What is the number of pairings in Carl's sense (when it matters who sits on which side of the board, but the boards are all alike), and in Diane's sense (when it is the other way around)?

1.1.3 What is the number of pairings (in all the various senses as above) in a party of 10?

1.2 Sets and the Like

We want to formalize assertions like "the problem of counting the number of hands in bridge is essentially the same as the problem of counting tickets in the lottery." The most basic tool in mathematics that helps here is the notion of a *set*. Any collection of distinct objects, called *elements*, is a set. The deck of cards is a set, whose elements are the cards. The participants in the party form a set, whose elements are Alice, Bob, Carl, Diane, Eve, Frank, and George (let us denote this set by P). Every lottery ticket of the type mentioned above contains a set of 5 numbers.

For mathematics, various sets of numbers are especially important: the set of real numbers, denoted by $\mathbb{R}$; the set of rational numbers, denoted by $\mathbb{Q}$; the set of integers, denote by $\mathbb{Z}$; the set of non-negative integers, denoted by $\mathbb{Z}_+$; the set of positive integers, denoted by $\mathbb{N}$. The *empty set*, the set with no elements, is another important (although not very interesting) set; it is denoted by $\emptyset$.

If A is a set and b is an element of A, we write $b \in A$. The number of elements of a set A (also called the *cardinality* of A) is denoted by $|A|$. Thus $|P| = 7$, $|\emptyset| = 0$, and $|\mathbb{Z}| = \infty$ (infinity).[1]

We may specify a set by listing its elements between braces; so

$$P = \{\text{Alice, Bob, Carl, Diane, Eve, Frank, George}\}$$

is the set of participants in Alice's birthday party, and

$$\{12, 23, 27, 33, 67\}$$

is the set of numbers on my uncle's lottery ticket. Sometimes, we replace the list by a verbal description, like

$$\{\text{Alice and her guests}\}.$$

Often, we specify a set by a property that singles out the elements from a large "universe" like that of all real numbers. We then write this property inside the braces, but after a colon. Thus

$$\{x \in \mathbb{Z} : \ x \geq 0\}$$

is the set of non-negative integers (which we have called $\mathbb{Z}_+$ before), and

$$\{x \in P : \ x \text{ is a girl}\} = \{\text{Alice, Diane, Eve}\}$$

(we will denote this set by G). Let us also tell you that

$$\{x \in P : \ x \text{ is over 21 years old}\} = \{\text{Alice, Carl, Frank}\}$$

(we will denote this set by D).

A set A is called a *subset* of a set B if every element of A is also an element of B. In other words, A consists of certain elements of B. We can allow A to consist of all elements of B (in which case $A = B$) or none of them (in which case $A = \emptyset$), and still consider it a subset. So the empty set is a subset of every set. The relation that A is a subset of B is denoted by $A \subseteq B$. For example, among the various sets of people considered above, $G \subseteq P$ and $D \subseteq P$. Among the sets of numbers, we have a long chain:

$$\emptyset \subseteq \mathbb{N} \subseteq \mathbb{Z}_+ \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

[1] In mathematics one can distinguish various levels of "infinity"; for example, one can distinguish between the cardinalities of $\mathbb{Z}$ and $\mathbb{R}$. This is the subject matter of *set theory* and does not concern us here.

The notation $A \subset B$ means that A is a subset of B but not all of B. In the chain above, we could replace the $\subseteq$ signs by $\subset$.

If we have two sets, we can define various other sets with their help. The *intersection* of two sets is the set consisting of those elements that are elements of both sets. The intersection of two sets A and B is denoted by $A \cap B$. For example, we have $G \cap D = \{\text{Alice}\}$. Two sets whose intersection is the empty set (in other words, they have no element in common) are called *disjoint.*

The *union* of two sets is the set consisting of those elements that are elements of at least one of the sets. The union of two sets A and B is denoted by $A \cup B$. For example, we have $G \cup D = \{\text{Alice, Carl, Diane, Eve, Frank}\}$.

The *difference* of two sets A and B is the set of elements that belong to A but not to B. The difference of two sets A and B is denoted by $A \setminus B$. For example, we have $G \setminus D = \{\text{Diane, Eve}\}$.

The *symmetric difference* of two sets A and B is the set of elements that belong to exactly one of A and B. The symmetric difference of two sets A and B is denoted by $A \triangle B$. For example, we have $G \triangle D = \{\text{Carl, Diane, Eve, Frank}\}$.

Intersection, union, and the two kinds of differences are similar to addition, multiplication, and subtraction. However, they are operations on *sets*, rather than operations on *numbers.* Just like operations on numbers, set operations obey many useful rules (identities). For example, for any three sets A, B, and C,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \tag{1.1}$$

To see that this is so, think of an element x that belongs to the set on the left-hand side. Then we have $x \in A$ and also $x \in B \cup C$. This latter assertion is the same as saying that either $x \in B$ or $x \in C$. If $x \in B$, then (since we also have $x \in C$) we have $x \in A \cap B$. If $x \in C$, then similarly we get $x \in A \cap C$. So we know that $x \in A \cap B$ or $x \in A \cap C$. By the definition of the union of two sets, this is the same as saying that $x \in (A \cap B) \cup (A \cap C)$.

Conversely, consider an element that belongs to the right-hand side. By the definition of union, this means that $x \in A \cap B$ or $x \in A \cap C$. In the first case, we have $x \in A$ and also $x \in B$. In the second, we get $x \in A$ and also $x \in C$. So in either case $x \in A$, and we either have $x \in B$ or $x \in C$, which implies that $x \in B \cup C$. But this means that $A \cap (B \cup C)$.

This kind of argument gets a bit boring, even though there is really nothing to it other than simple logic. One trouble with it is that it is so lengthy that it is easy to make an error in it. There is a nice graphic way to support such arguments. We represent the sets A, B, and C by three overlapping circles (Figure 1.1). We imagine that the common elements of A, B, and C are put in the common part of the three circles; those elements of A that are also in B but not in C are put in the common part of circles A and B outside C, etc. This drawing is called the *Venn diagram* of the three sets.

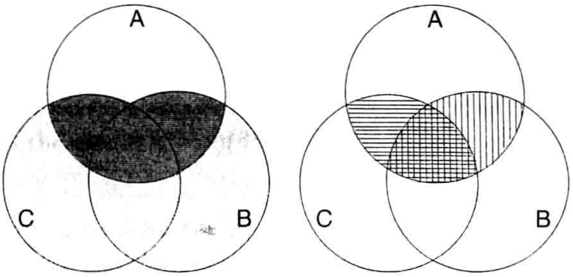

FIGURE 1.1. The Venn diagram of three sets, and the sets on both sides of (1.1).

Now, where are those elements in the Venn diagram that belong to the left-hand side of (1.1)? We have to form the union of B and C, which is the gray set in Figure 1.1(a), and then intersect it with A, to get the dark gray part. To get the set on the right-hand side, we have to form the sets $A \cap B$ and $A \cap C$ (marked by vertical and horizontal lines, respectively in Figure 1.1(b)), and then form their union. It is clear from the picture that we get the same set. This illustrates that Venn diagrams provide a safe and easy way to prove such identities involving set operations.

The identity (1.1) is nice and quite easy to remember: If we think of "union" as a sort of addition (this is quite natural), and "intersection" as a sort of multiplication (hmm... not so clear why; perhaps after we learn about probability in Chapter 5 you'll see it), then we see that (1.1) is completely analogous to the familiar distributive rule for numbers:

$$a(b+c) = ab + ac.$$

Does this analogy go any further? Let's think of other properties of addition and multiplication. Two important properties are that they are *commutative*,

$$a + b = b + a, \qquad ab = ba,$$

and *associative*,

$$(a+b)+c = a+(b+c), \qquad (ab)c = a(bc).$$

It turns out that these are also properties of the union and intersection operations:

$$A \cup B = B \cup A, \qquad A \cap B = B \cap A, \tag{1.2}$$

and

$$(A \cup B) \cup C = A \cup (B \cup C), \qquad (A \cap B) \cap C = A \cap (B \cap C). \tag{1.3}$$

The proof of these identities is left to the reader as an exercise.

Warning! Before going too far with this analogy, let us point out that there is another distributive law for sets:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \tag{1.4}$$

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