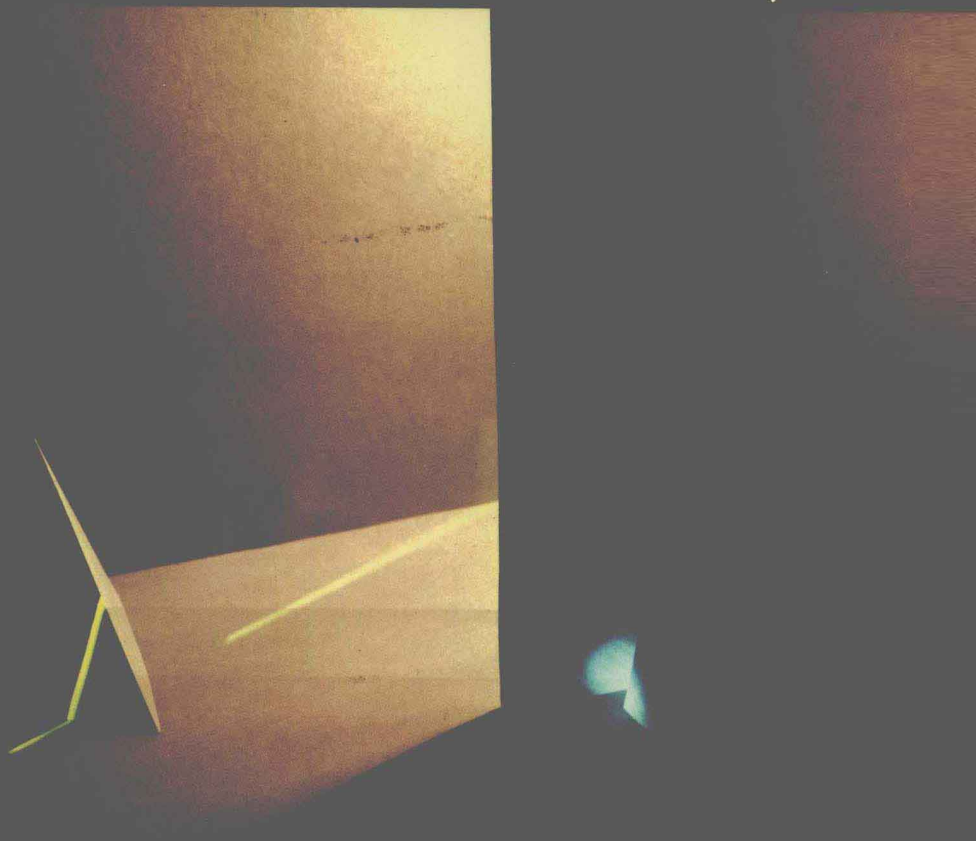
COLLEGE GEOMETRY

A Problem - Solving Approach with Applications



Gary L. Musser

Lynn E. Trimpe

College Geometry

A Problem-Solving Approach with Applications

Gary L. Musser
Oregon State University

Lynn E. Trimpe
Linn-Benton Community College

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To my wife, Irene, to my son, Greg,
to my mother, Marge, and to my father, G.L.
—Gary

To my mother and father, Shirley and Howard,
and to my husband, Tim
—Lynn

Special Dedication to William F. Burger

This book is dedicated to Bill, Gary's friend and close associate at Oregon State University, who died prematurely from pancreatic cancer on March 23, 1991. Bill and Gary had taught together for over twelve years and coauthored *Mathematics for Elementary Teachers—A Contemporary Approach,* published by Macmillan.

Shortly after coming to Oregon State, Bill was the director of a research project funded by the National Science Foundation that studied how children learn geometry, especially with respect to the van Hiele levels. His pioneering work brought him international recognition in this area. Bill's passion for geometry led to the idea of writing this book. In the planning stage of this book and shortly before Bill passed away, Lynn was asked to join our team because of her expert teaching at the community college level and her valued contributions to the supplements for the Musser/Burger book for elementary teachers.

In addition to Bill's notable work in geometry, he received the OSU College of Science Carter Award for outstanding and inspirational teaching, he founded MAJIC, the Math Advising for Juniors Interested in College placement test for the state of Oregon, and he was a key participant in the OSU SMILE program, which encouraged science and mathematics achievement among Native American and Hispanic students in Oregon's rural schools. He spoke and published widely and was active in professional organizations. Most of all, he was revered by his many students at OSU.

In his personal life, Bill played oboe in a local woodwind quintet, was an active member in his church, loved opera, was a devoted fan of the Cleveland Indians since boyhood, played softball until he broke his ankle sliding into second base, enjoyed his push-button transmission Chrysler Newport with a square steering wheel, and treasured the time spent with his wife, Adrienne, and his precious daughter, Mary.

Bill was a very special person. It was a joy working with him. He had a feel for and love of geometry that is seldom seen, and his love of teaching and his students was inspiring. He is missed by all who were touched by him. Although this book would have no doubt been richer had he been here to work with us, we believe that its spirit would please him.

List of Supplements

Student Activity Manual and Study Guide
Instructor's Manual
Computerized Test Bank

Preface

PHILOSOPHY

This book has four main goals:

1. To help students become better problem solvers, especially in solving common application problems involving geometry
2. To help students learn many properties of geometric figures, to verify them using proofs, and to use them to solve applied problems
3. To expose students to the axiomatic method of synthetic Euclidean geometry at an appropriate level of sophistication
4. To provide students with other methods for solving problems in geometry, namely using coordinate geometry and transformation geometry.

To accomplish these goals, we have organized the text into three parts:

Part I—Problem Solving, Geometric Shapes, and Measurement (Chapters 1–3)

Part II—Formal Synthetic Euclidean Geometry (Chapters 4–7)

Part III—Alternate Approaches to Plane Geometry (Chapters 8–9).

Our rationale for this arrangement takes into account the students who will study this book. First, research on learning geometry suggests that the study of geometry should begin with informal experiences and gradually move toward formal proof. Second, most of these students will be using geometry to solve problems encountered in their future vocations as well as in subsequent coursework. Since the notions of geometric shapes and measurement geometry are particularly relevant to these students, we provide an early development of these ideas in an informal manner. In this way, many ideas that are proved in Part II are introduced informally and some are justified intuitively in Part I.

Part II provides the core of a standard Euclidean geometry course. We have postulated some results that could be theorems. This allows the students to get to the central results more quickly, giving them additional time to apply the results to other proofs and applied problems.

Part III opens students' eyes to the fact that there are other ways to prove results and solve problems in geometry. Students grow to appreciate the beauty and efficiency of using these alternate approaches.

FEATURES

The following features have been incorporated into the design of this book to enhance student learning.

Pedagogical

- Two-color format
- A liberal use of examples throughout
- Extensive use of figures, many in two colors, to enhance the concept development
- Boxes to highlight postulates, theorems, and important definitions
- Bold face type to highlight definitions
- Many theorems motivated by first considering specific examples
- A distinctive symbol to indicate the end of an example or a proof
- Problem sets that include various combinations of exercises, problems, applied problems, and proofs
- Answers for all odd-numbered exercises and problems
- Answers for odd-numbered proofs either outlined, or given in a paragraph or statement-reason format
- Problem-solving strategies and clues given in each chapter together with additional problems that utilize the highlighted strategy
- Writing for Understanding problems to give students an opportunity to deepen their understanding and to communicate in writing
- Chapter Reviews that require active student participation and self-assessment
- Chapter Tests that serve as a review of students' abilities to work with basic concepts, solve problems, and make proofs
- A brief Table of Contents and a second more complete one listing every section
- Common geometric formulas with figures, a symbol list, and table of conversions on the inside cover pages
- Topics Sections at the end of the book that provide some prerequisite material and some enrichment topics
- Appendix 1, Getting Started, to which students can refer when starting proofs
- Appendix 2, a quick reference of all the postulates, theorems, and corollaries

Motivational

- Chapter openers that set the scene for each chapter by presenting a historical tidbit
- Initial applied problems that motivate the material in each section and that have a solution given at the end of the section

- Geometry Around Us features at the end of each section that make students cognizant of examples of geometry in our world
- Vignettes about People in Geometry that acquaint students with some well-known geometers

SUPPLEMENTS

Student Activity Manual and Study Guide—This resource contains many hands-on activities correlated with chapters in the text to promote concept learning, solutions for every other odd-numbered problem, hints and written solutions for selected proofs, complete solutions to chapter tests, and additional practice tests with answers.

Instructor's Manual—This manual contains answers to all even-numbered exercises, problems, and applied problems, with outlines of all even-numbered proofs, overhead transparency masters, two chapter tests for each chapter and one for each topic section, and a listing of all questions that appear in the computerized test bank, and some problems for computer exploration.

Computerized Test Bank—All of the test items in the Instructor's Manual are available on disk, both in MS-DOS and in Macintosh™ formats.

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We also thank the many students who have used our preliminary version of the text and given us encouragement through their success. Last we thank our production team, especially Kelly Ricci and Mercedes Jackson at Spectrum Publisher Services and Aliza Greenblatt at Macmillan, for bringing this book to print and our visionary editor, Bob Pirtle, for his smooth coordination of this project.

G.L.M.

L.E.T.

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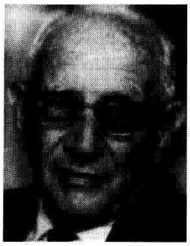
CHAPTER

1

Problem Solving in Geometry



George Polya (1887–1985) was a mathematician famous for his lifelong interest in and study of the process of problem solving. Born and educated in Hungary, Polya came to the United States in 1940. He is the author of numerous books and papers on problem solving, the most famous of which is *How to Solve It.* This book has been translated into 17 different languages, and more than 1,000,000 copies have been sold since it was first published in 1945.

Polya defined intelligence as the ability to solve problems and believed that "solving problems is human nature itself." Thus he felt strongly that a major goal of education should be the development of problem-solving skills. To that end, Polya devised what has become known as the four-step problem-solving process, which will be a focus of this chapter.

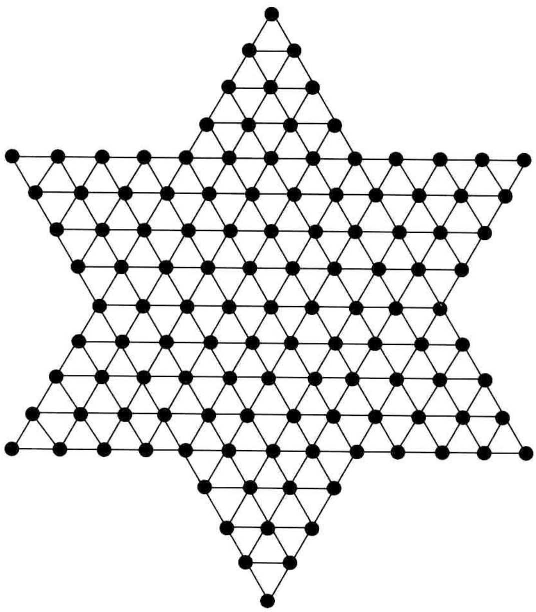
PROBLEM-SOLVING STRATEGIES

1. *Draw a Picture*
2. *Guess and Test*
3. *Use a Variable*
4. *Look for a Pattern*
5. *Make a Table*
6. *Solve a Simpler Problem*

One of the main goals of this book is to help you to become a better problem solver. This chapter introduces six strategies that will be useful in solving geometry problems. In addition, at the beginning of each of the remaining chapters, a new strategy is introduced. In this way, your ability to solve problems should grow much as the Problem-Solving Strategies boxes like the one to the left grow throughout the book.

INITIAL PROBLEM

Chinese checkers is a marble game played by 2 to 6 players on a board like the one shown below. This board has a starting pen (triangular region in one point of the star) with 4 rows of holes that holds 10 marbles for each player. Suppose that a more challenging game is desired and a similar board is constructed to have 6 rows (21 marbles) for each starting pen. How many holes, in all, would this expanded board have?



INTRODUCTION

Problem solving is considered by most people to be the main objective of any mathematics course. In a more general sense, problem solving is central to many disciplines, including business and engineering. This chapter provides an introduction to the study of problem solving with an emphasis on problems in geometry.

1.1 PROBLEM-SOLVING STRATEGIES

In our discussion of problem solving, we need to make clear what is meant by a problem. In this book a distinction is made between an "exercise" and a "problem." An **exercise** can be solved by applying a routine procedure. It may be similar to other exercises (or problems) that you have worked or have seen worked. A typical "word problem" in an algebra course is an exercise if you recognize the type of problem and can recall an appropriate procedure to apply.

On the other hand, a **problem** is nonroutine and unfamiliar. To solve a problem, you need to stop and think about how to attack it. You may need to try something completely new and different. You may get stuck several times and have to make new starts. The need for some kind of creative step on your part in solving the problem is what makes it different from an exercise.

George Polya was a mathematician whose name has come to be synonymous with problem solving. In an effort to encourage more students to "experience the tension and enjoy the triumph of discovery" that accompany solving a problem, he presented the following **four-step process** for problem solving.

STEP 1. Understand the Problem.

- Is it clear to you what is to be found?
- Do you understand the terminology used in the problem?
- Is there enough information?
- Is there irrelevant information?
- Are there any restrictions or special conditions to be considered?

STEP 2. Devise a Plan.

- How should the problem be approached?
- Does the problem appear similar to any others you have solved?
- What strategy might you use to solve the problem?

STEP 3. Carry Out the Plan.

- Apply the strategy or course of action chosen in Step 2 until a solution is found or you decide to try another strategy.

STEP 4. Look Back.

- Is your solution correct?
- Do you see another way to solve the problem?
- Can your results be extended to a more general case?

In the four-step process, Step 2 is a critical one. Even if you thoroughly understand a problem (Step 1), you may not be able to progress further. On the other hand, once you have selected a workable strategy, it is usually not difficult to implement it (Step 3). Likewise, once a solution has been obtained, it is usually not difficult to verify whether that solution is correct (Step 4). Therefore, the purpose of this chapter is to focus on Step 2 in Polya's process and to present six general strategies that are frequently useful in solving geometric problems. Other strategies will be presented in subsequent chapters of the book, one strategy at the beginning of each chapter.

STRATEGY 1: DRAW A PICTURE

This strategy is a natural choice for solving problems in geometry because many of the problems of geometry are related to figures, shapes, and physical structures. Often drawing one or several pictures can help you to solve the problem or can help you to better understand the problem so that you can formulate a plan for solving it. The following clues may help you to identify situations where the Draw a Picture strategy might be useful.

CLUES

The Draw a Picture strategy may be appropriate when

- A physical situation is involved.
- Geometric figures or measurements are involved.
- You want to gain a better understanding of the problem.
- A visual representation of the problem is possible.

As you attempt to solve the following example problems, imagine solving the problems *without* looking at any pictures. Then try solving the problems with a picture to see if the picture helped in the process.

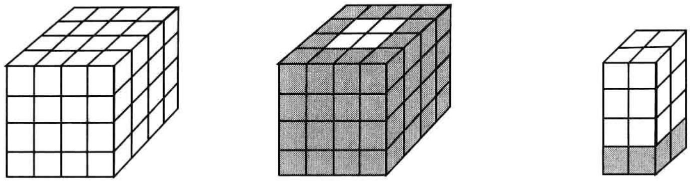
EXAMPLE 1.1 A large cube is formed by arranging 64 smaller cubes of the same size in a stack that measures 4 cubes by 4 cubes by 4 cubes. An open cardboard box is constructed in the shape of the larger cube, and the arrangement of small cubes just fits into the box. How many of the small cubes are *not* touching a side or the bottom of the box?

STEP 1. Understand the Problem.

There are no gaps between the small cubes and the sides of the box. Remember that the box has no top. We must determine how many of the 64 small cubes are in contact with neither a side nor the bottom of the box.

STEP 2. Devise a Plan.

Draw a picture of the cubes inside the box in order to better visualize their arrangement [Figure 1.1(a)]. The 4 by 4 by 4 cube has 64 small cubes in it, arranged in four layers.



(a) (b) (c)

FIGURE 1.1

STEP 3. Carry Out the Plan.

Now we can start counting the small cubes that touch the sides and bottom of the box. Shading in those cubes that are in contact with a side or the bottom of the box may be helpful [Figure 1.1(b)]. Remove the central "core" of cubes [Figure 1.1(c)]. Some of these cubes touch only the bottom of the box. We can see that the top 3 layers of 4 cubes each, or 12 cubes in all, touch neither the sides nor the bottom of the box.

STEP 4. Look Back.

We also could have solved this problem by subtracting the number of cubes that *do* touch the sides and bottom of the box from 64. What would happen if the large cube measured 5 by 5 by 5? How many small cubes would touch neither the sides nor the bottom of that cube? Can this problem be generalized to an n by n by n cube? If so, how?

Additional Problems Where the Strategy "Draw a Picture" Is Useful

1. If the diagonals of a square are drawn in, how many triangles of all sizes are formed?

To Get You Started

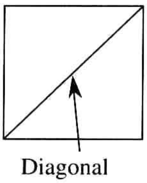


FIGURE 1.2

   (a) Do you know what the diagonal of a square is? It is a line segment whose endpoints are two opposite corners of the square. One diagonal is shown in Figure 1.2. (A more general definition of a diagonal will be given later in the book.)
   (b) Remember that the triangles may be different sizes. Also, some of the triangles may overlap.
   (c) Although one picture may be helpful in formulating a solution to the problem, several pictures might make the solution even clearer.

2. A rectangular milk crate has spaces for 24 bottles in 4 rows and 6 columns. Can you put 18 bottles of milk into the crate so that each row and each