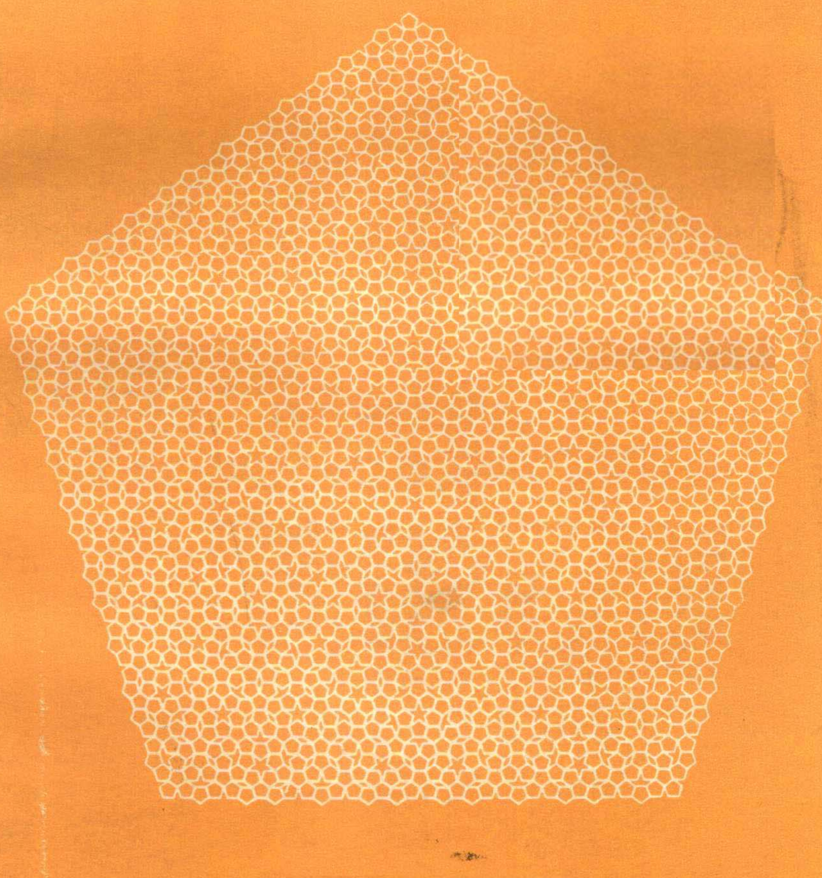
OXFORD LOGIC GUIDES

Choice Sequences

A Chapter of Intuitionistic Mathematics

A. S. TROELSTRA



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For Jane Bridge

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PREFACE

These notes on choice sequences originated in a series of lectures given at the Mathematical Institute at Oxford during Michaelmas Term 1973. The aim of these lectures was to introduce choice sequences informally, stressing the concepts, not the purely formal axiomatic treatment. Applications and metamathematical results which might be helpful in understanding the notions were also described, but as a rule not proved, since most of the relevant literature is easily accessible.

However, in preparing the notes of these lectures for publication, it seemed useful not only to describe the relevant metamathematical results, but to include sketches of proofs as well, so as to give the reader an idea of the methods, and to facilitate the study of detailed proofs in the existing literature.

In some cases, the treatment in the literature was defective, in other cases unnecessarily formal, thereby obscuring the underlying ideas. Thus I have included (1) a rather detailed discussion of the elimination of lawless sequences (Chapter 3); (2) a simpler and more informal presentation of the continuity properties of the universe $\mathscr{U}$ and details on the connection between topological models and validity in universes $\mathscr{U}_\alpha$ (Chapter 4); (3) a modernized presentation of the main result of Dyson and Kreisel (1961) (Chapter 7).

As a result, these notes constitute a fairly comprehensive introduction to the topic of choice sequences. Chapters 1-7 contain 'what everybody with a serious interest in intuitionism should know about choice sequences'; the three appendices are devoted to historical remarks (including some discussion of the literature), illustrations of certain aspects, and 'open ends' (i.e. unfinished or inconclusive developments) of the subject which might stimulate further research.

As to the prerequisites I shall assume some general logical background (such as first order predicate logic with the completeness theorem, some basic notions from recursion theory) which may be gleaned from any not-too-elementary standard text (e.g. Kleene's Mathematical Logic), and in addition, some familiarity with the basic principles of intuitionism, such as can be obtained from any of the following sources: Heyting (1956), Chapters I-III, VII; Troelstra (1969), sections 1-8; van Dalen (1973), sections 0-2, or Dummett (1976), Chapters I-II.

Acknowledgements.

Various people have helped me by critical comments, lists of misprints etc.; I wish to mention here especially G. Kreisel, R. C. de Vrijer, and my wife. I am indebted to H. de Swart for his permission to make use of his unpublished work on the completeness of intuitionistic predicate logic for Chapter 7. My assistants G. F. van der Hoeven, G. Renardel, and J. J. Willemen assisted me in the tedious task of proof reading.

It was Jane Bridge who first suggested that the notes of my lectures on choice sequences might be suitable for publication in the Oxford Logic Guides. Her friendship greatly contributed to the enjoyment of my family and myself during our stay in Oxford, and therefore I am happy to be able to dedicate these notes to her.

Muiderberg,
August 1975. A.S.T.

CONTENTS

1

INTRODUCTORY REMARKS; PRELIMINARIES; LAWLIKE OBJECTS

1.1.

In these notes, we shall adopt the intuitionistic viewpoint, not as a philosophy of mathematics that excludes others, but as the appropriate framework for describing a *part* of mathematical experience.† That is to say, we shall be dealing with concepts (choice sequences) which are most naturally treated intuitionistically.

Our interpretation of 'intuitionistic' implies that we shall adopt the subjectivistic view†† of mathematical truth: what is true is, for the idealized mathematician, what he can establish for himself by (mental) reflection about his own constructions (which include in particular those reflections (proofs)). Mathematical language is secondary; in particular, mathematical objects are not necessarily presented to us in a linguistic framework. In this respect our view agrees with that of classical set theory or geometry, where not every set of natural numbers nor every point in the plane is assumed to be definable (in a given linguistic framework). 'Secondary' means here only that the use of lan-

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†The view that various parts of mathematical experience (in the widest sense) correspond to different views in the philosophy of mathematics is clearly expressed and discussed in Kreisel (1965), pages 95-98, 184-192.

††We use 'subjectivistic', in contrast to 'objectivistic', as referring to a concept of mathematical truth, as in platonism, which is independent of human knowledge of the truth; not in contrast to 'intersubjective' (in the sense of being valid for all mathematicians), since the 'idealized mathematician' is an idealization of any individual mathematician. But cf. also the remark on the theory of the 'creative subject' below.

guage is *mathematically* irrelevant to our objects of study, not that it is generally unimportant: in practice, language is an indispensable tool, not only because we ourselves are not idealized mathematicians with unlimited memory, but also because many special classes of operations are introduced by definability conditions (schemata).

It should be stressed at this point that talking about the 'idealized mathematician' in no way commits us to adopt the speculative features of Brouwer's theory of the 'creative subject' (the idealized mathematician); that is, first the possibility of explicit reference to the course of mathematical activity of the idealized mathematician in constructions by the idealized mathematician, and secondly the division of *all* mathematical activity into ω stages. The idealization in the concept of 'idealized mathematician' is of the same sort as that required by a platonist philosophy of mathematical knowledge, where we also assume e.g. the possibility of undistorted and immediate insight into the cumulative hierarchy of sets.

1.2. Lawlike objects.

A mathematical object will be called "lawlike" if it is a completed construction, something we can describe (to ourselves) completely. In some publications the word 'constructive' is used instead of lawlike (e.g. in Dragalin 1973). The various concepts of choice sequence will constitute an extension of the domain of (lawlike) mathematical objects; as will be clear from our discussions in Chapter 2 and afterwards, they cannot be considered to be completed constructions.

The two principal kinds of lawlike objects we have to consider are natural numbers and lawlike functions of natural numbers (with natural numbers as values). There is no need to say much about the natural numbers - but let us consider the idea of a lawlike function of natural numbers somewhat more closely. A lawlike function (of the type $N \to N$, N denoting the natural numbers) should be a law ('recipe') determining in an effective way a value for each argument; the law is a completely described mathematical object. Such a law is given to us in a way that makes it clear that the law is applicable to each natural number, i.e. a proof that the value is always defined is implicit. So a lawlike function can certainly not be specified by simply presenting its graph - that is a far too platonistic idea. But for example presenting a gödelnumber is not enough either - we need to know that the gödelnumber is the number of a total function, so a proof of this fact must then be supposed to be appended. Primitive recursive functions, presented in the standard way, are obviously total, and hence lawlike.

Apart from natural numbers and lawlike functions of natural numbers, there are two other categories of lawlike objects which we shall enoounter: species (sets of relations) and lawlike operations (functionals) defined on various universes of sequences (e.g. defined on all lawless sequences) and taking natural numbers or other sequences as values; this type of lawlike object will be discussed at length later on.

In these notes we do not commit ourselves as to which instances of the comprehension schema we consider acceptable intuitionistically - the discussion of the theory of choice sequences is to a large extent independent of the comprehension axioms adopted.

1.3. Church's thesis for lawlike functions.

Should we accept the intuitionistic form of Church's thesis, i.e. the statement

'Every lawlike function is recursive'?

Or expressed formally†:

CT $\quad \forall a \exists x \forall y \exists z (Txyz \ \& \ Uz = ay)$

(where a is a variable ranging over lawlike sequences, x, y, z variables ranging over N, T is Kleene's T-predicate, and U the result-extracting function as in Kleene 1952).

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†The quantifiers are of course interpreted intuitionistically!

There are two reasons for abstaining from the identification 'lawlike = recursive':

(i) An axiomatic reason: the developments in the sequel do not depend on this identification - therefore explicitly assuming recursiveness means carrying unnecessary information around. In the formal developments, there are many possible interpretations for the range of the variables for lawlike sequences (e.g. the classical universe of sequences).

(ii) A second reason is 'philosophical': the (known) informal justifications of 'Church's thesis' all go back to Turing's conceptual analysis (or proceed along similar lines).

Turing's analysis strikes me as providing very convincing arguments for identifying 'mechanically computable' with 'recursive', but as to the identification of 'humanly computable' with 'recursive', extra assumptions are necessary which are certainly not obviously implicit in the intuitionistic (languageless) approach as adopted here. See also Gödel's remarks on pages 72-73 of Davis (1965).

1.4. *Intensional and extensional aspects.*

In Troelstra (1975) I have discussed at some length the distinction 'extensional-intensional'; there is no need to repeat the discussion here in full. For our purposes, the following will suffice: 'extensionally' in connection with functions means referring to the *graph* of the function alone. In other words, if α,β are functions, 'α is extensionally equal to β' (notation $\alpha = \beta$) is defined by

$$\alpha = \beta \equiv_{\mathrm{def}} \forall x(\alpha x = \beta x)$$

and a predicate A (of functions) is said to be extensional if

$$\forall\alpha\forall\beta(\alpha = \beta \;\&\; A\alpha \to A\beta).$$

But of course, functions are *given* to us in some way - which means that we actually have more information about a function than just its graph. For example the gödelnumber of a recursive function b cannot be determined from the graph of b alone - although when b is given to us as recursive, the available information must permit us to find a gödelnumber for b. Another example, from classical mathematics, is provided by the treatment of functions in category theory, which are supposed to be specified with domain and codomain, not simply by their graph.

We shall loosely refer to this extra information as 'the intensional aspects', and use phrases like 'speaking intensionally', meaning: 'with reference to the intensional aspects'.

Intensional equality in the strict sense is really identity: two objects are intensionally equal if, and only if, they are given to us as the same object.

1.5. Axioms of choice, selection principles.

The following 'axiom of choice' AC_{00} or AC-NN

AC-NN $\quad \forall x\exists yA(x,y) \to \exists a\forall xA(x,ax)$

(x,y numerical variables, a,b ranging over lawlike functions = lawlike sequences) is almost more logical than mathematical in character, i.e. it follows from the intended meaning of the logical operations: a proof of $\forall x\exists yA(x,y)$ should contain a method ('rule', 'law', 'recipe') for constructing a y to each x - but such a method is nothing else but a lawlike function.

Somewhat more generally, we have the schema AC_{01} or AC-NF:

AC-NF $\quad \forall x\exists aA(x,a) \to \exists b\forall xA(x,(b)_x)$

where $(b)_x = \lambda y.bj(x,y)$, j is a pairing function onto the natural numbers with j_1,j_2 its inverses, and where A is supposed to be extensional in the function parameter:

$$A(x,a) \;\&\; a = a' \to A(x,a').$$

The justification is similar: a proof of $\forall x\exists aA(x,a)$ contains a method for finding an a for each x; now define b by: to compute bz, take j_1z, find the a from the method given by the proof of $A(j_1z,a)$, and let $bz = a(j_2z)$.

We should *not* expect (even for extensional A)

(1) $\quad \forall a\exists xA(a,x) \to \exists\Gamma\forall aA(a,\Gamma a)$

to hold (Γ an operator from functions to natural numbers) when we require Γ to be a functional in the classical extensional sense, i.e. satisfying

$$a = b \to \Gamma a = \Gamma b.$$

On the assumption of CT it is quite easy to see why (1) is implausible if Γ is to be extensional: take $\forall y \exists z(Txyz \;\&\; Uz = ay)$ for $A(a,x)$, then (1) requires an *extensional* Γ assigning a gödelnumber to each recursive a. Then Γ cannot possible be represented by a partial recursive operation defined on all gödelnumbers of total recursive functions. There is an obvious solution for Γ, namely the 'identity' function (i.e. assigning x to the recursive function with gödelnumber x), but this solution is obviously not extensional.

The only fact which is clearly implicit in the intended interpretation of the quantifier-combination $\forall a \exists b$ is that there is some operator Ψ such that $\forall a A(a, \Psi a)$; but Ψ may depend for its evaluation not only on the graph of a but also on intensional aspects of a.

For example, let $\mathfrak{A}$ range over a definable set $X = \{x: Bx\}$ of natural numbers, and y over the natural numbers N. The 'natural' equality relation on X is the equal ty induced by equality on N (i.e. equality between natural numbers); but elements of X are given to us as such by a pair (n,p), $n \in N$, p a proof of Bn. Now $\forall \mathfrak{A} \exists y A(\mathfrak{A}, y)$ becomes $\forall x(Bx \to \exists y A(x,y))$; and on the interpretation of the logical constants we see that this implies that y can be found depending on x *and* on a proof of Bx; the method for finding y is not 'extensional in x'.

Quite generally, we shall refer to principles of the form

$$\forall \mathfrak{A} \exists \mathfrak{B} A(\mathfrak{A}, \mathfrak{B}) \to \exists \Psi \forall \mathfrak{A} A(\mathfrak{A}, \Psi \mathfrak{A})$$

as 'selection principles'; it is understood that Ψ need not be extensional.

1.6. Convention.

We shall often talk about 'lawlike' sequences meaning sequences extensionally equal to lawlike ones (but not necessarily given as a lawlike sequence). This will not be a prolem as most of the time we shall discuss extensional properties of sequences only.

1.7. Choice sequences as a generalization of the concept of sequence.

Lawlike sequences are completely determined by a law given in advance. What is essential for mathematical purposes however, is that the concept of sequence is such that to each argument a value can be determined. The various concepts of choice sequence generalize the concept of lawlike function such that the idea of being determined by a law given in advance is abandoned, but the essential property of a value existing for each argument is retained.

The simplest concept of choice sequence is that of a lawless sequence, which will be discussed extensively in Chapter 2. For lawless sequences we can present an axiomatization completely characterizing their properties relative to a theory of lawlike objects; the axiomatization is obtained by a very convincing conceptual analysis. For the philosophy of mathematics, the possibility of such an analysis seems to us to be of considerable interest; and this interest is not changed by the fact that the results show that quantification over lawless sequences may be viewed as a 'manner of speech'.

1.8. Description of EL *and* EL_1.

EL (EL = elementary analysis) contains variables for natural numbers: (x,y,z,u,v,w,n,m) and for (lawlike) functions (a,b,c,d); constants: 0 (zero), S (successor), $=$ (equality between natural numbers), λ (abstraction operator for the introduction of functions by explicit definition), R (recursor, an operation for definition by recursion), j, j_1, j_2 (a pairing function onto the natural numbers with inverses), and Φ (for application of functions to numbers); the logical constants are $\&$, $\vee$, $\rightarrow$, $\forall$, $\exists$ (for functions and numbers).

Below we shall use A, B, C, D as syntactical variables for formulae, t, s as syntactical variables for numerical terms, and ϕ, ψ, ξ for functors (functional terms). $\neg A$ is an abbreviation of $A \to S0 = 0$; logical equivalence $\leftrightarrow$ is also treated as defined. ϕt abbreviates $\Phi\phi t$. Other conventions and abbreviations which will not be explained are standard (e.g. 1, 2, 3, ... for $S0$, $SS0$, $SSS0$, ...).

To indicate parameters in terms and functors we use square brackets; e.g. $t[x]$ indicates a term with (numerical) parameter x. Once $t[x]$ has been introduced, $t[t']$ indicates the term obtained by substitution of t' for x in t; a more accurate, but also more cumbersome notation is $t_x[t']$.

EL is based on two-sorted intuitionistic predicate logic, contains the usual axioms for successor and equality, pairing axioms:

$$j_1 j(x,y) = x,\ j_2 j(x,y) = y,\ j(j_1 z, j_2 z) = z,$$

induction with respect to all formulae in the language, the conversion rule

$$(\lambda x.t[x])t' = t[t']$$

(possibly renaming bound variables in t so as to avoid clashes of variables), the axioms for primitive recursion

$$\begin{cases} Rxa0 = x \\ Rxa(Sy) = aj(Rxay, y) \end{cases}$$

and a very weak axiom of choice

QF-AC $\quad \forall x \exists y A(x,y) \to \exists a \forall x A(x, ax) \quad$ (A quantifier-free).

In the presence of primitive recursive functions, QF-AC expresses closure under 'recursive in' for the domain of functions, therefore the minimal model for QF-AC consists of the natural numbers and all recursive functions over the natural numbers.

EL_1 is the system obtained from EL by replacing QF-AC by

AC-NF $\quad \forall x \exists a A(x,a) \to \exists b \forall x A(x, (b)_x)$

where, as before,

$$(b)_x \equiv_{def} \lambda y.bj(x,y).$$

1.9. Other notations and conventions (for consultation when needed).

For future use we introduce here some other notations and conventions.

(A) For coding u-tuples we use ν_u, with inverses $j^u_1, j^u_2, \dots, j^u_u$:

$$\nu_u(j^u_1 x, \dots, j^u_u x) = x,\ j^u_i \nu_u(x_1, \dots, x_u) = x_i.$$

(For the sake of definiteness, we may take e.g.
$\nu_1(x) = x$, $\nu_2(x,y) = j(x,y)$, $\nu_{u+1}(x_1, \dots, x_{u+1}) = j(\nu_u(x_1, \dots, x_u), x_{u+1})$.)
Also $b(x_1, \dots, x_u)$ stands for $b\nu_u(x_1, \dots, x_u)$.

(B) We assume a (primitive recursive) coding of all finite sequences of natural numbers *onto* the natural numbers to be given; in talking about sequences, we usually shall not distinguish between the sequence and its coding. Let $\langle x_0, \dots, x_u\rangle$ be the (code number of) the sequence $x_0, \dots, x_u$; $*$ indicates *concatenation*, i.e.

$$\langle x_0, \dots, x_u\rangle * \langle x_{u+1}, \dots, x_v\rangle = \langle x_0, \dots, x_v\rangle.$$

We shall assume

$$\langle\rangle = 0.$$

We put

$$n \preceq m \equiv_{\text{def}} \exists n'(n * n' = m)$$
$$n \prec m \equiv_{\text{def}} n \preceq m \ \&\ n \neq m.$$

The *length-function* lth satisfies

$$\begin{cases} \text{lth}\,\langle\rangle = \\ \text{lth}\,\langle x_0, \dots, x_u\rangle = u+1. \end{cases}$$

For the *inverse* to sequence-coding we write $(n)_x$, i.e. if $n = \langle x_0, \dots, x_u\rangle$, then

$$(n)_y = x_y \text{ for } y \le u,\ (n)_y = 0 \text{ elsewhere.}$$

We introduce the abbreviation

$$\hat{n} \equiv \langle n\rangle$$

and

$$\begin{cases} \bar{a}0 \equiv \langle\rangle \\ \bar{a}u \equiv \langle a0, \dots, a(u \dot{-} 1)\rangle \end{cases}$$

(where $x \dot{-} y$ is defined as usual by $x \dot{-} 0 = x$, $x \dot{-} Sy = \text{prd}(x \dot{-} y)$, and prd (predecessor) satisfying $\text{prd}\ 0 = 0$, $\text{prd}(Sx) = x$). Note that $\bar{\alpha}(u+1) = \bar{\alpha}u * \langle \alpha u\rangle$.