Unified Transform for Boundary Value Problems

Applications and Advances

Edited by
A. S. Fokas • B. Pelloni

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Preface

In May 2012, the International Centre for Mathematics Sciences (ICMS) in Edinburgh hosted the workshop *Boundary Value Problems for Linear Elliptic and Integrable PDEs: Theory and Computation*. This workshop brought together a small group of mathematicians interested, from several different perspectives, in the solution of boundary value problems, namely the type of problems modeled by partial differential equations (PDEs) that are most commonly arising in applications.

This meeting focused in particular on the applications of the so-called *unified transform* (also referred to as the *Fokas transform* or the *method of Fokas*) to the analysis and numerical modeling of boundary value problems for linear and integrable nonlinear PDEs and on the closely related *boundary element method*, a well-established numerical approach for solving linear elliptic PDEs. The latter method can be viewed as the counterpart in the physical space of the numerical implementation of the unified transform, which is formulated in the spectral (Fourier) space.

This book was conceived during the workshop mentioned above and collects the results of the exchanges of ideas fostered by the meeting. The chapters are closely related and, when put together, paint a picture of the state of the art in the advances and applications of the unified transform as well as its relation with the boundary element method.

It is divided in three main parts. Part I contains new theoretical results on evolution and elliptic problems, linear and nonlinear. New explicit solution representations for several classes of boundary value problems are constructed and rigorously analyzed.

Part II, at the center of the book, is a detailed overview of variational formulations for elliptic problems, building up to placing the unified transform approach in this classical context, alongside the boundary element method, and stressing its novelty.

Part III presents recent numerical applications based on the boundary element method and on the unified transform.

Contents

Chapter 1
Introduction

A. S. Fokas and B. Pelloni

1.1 ▪ Boundary Value Problems

In 1747, d'Alembert derived the wave equation, which was the first partial differential equation (PDE) in the history of mathematics. Soon after, d'Alembert and Euler discovered a general method for constructing large classes of solutions, namely the method of *separation of variables*. Bernoulli introduced the infinite sine series and Euler discovered the standard formula for the coefficients of a Fourier series. Fourier inaugurated in 1807 the era of linearization. In 1814 Cauchy wrote as essay on using complex variables for the evaluation of certain integrals. In 1828 Green introduced the powerful approach of integral representations that can be obtained via Green's functions. Separation of variables led to the *spectral analysis* of ordinary differential operators and to the solution of PDEs via a *transform pair*. The prototypical such pair is the Fourier transform; variations include the sine, the cosine, and the Laplace and Mellin transforms.

In the second half of the 20th century it was realized that certain nonlinear evolution PDEs, called *integrable*, can be formulated as the compatibility condition of two linear eigenvalue equations called a *Lax pair* and that this formulation gives rise to a method for solving the initial value problem for these equations. This method is now known as the *inverse scattering transform* method. One of us has emphasized that this method is based on a deeper form of separation of variables [1]. Indeed, the spectral analysis of the spatial part of the Lax pair yields an appropriate *nonlinear Fourier transform pair*, whereas the temporal part of the Lax pair yields the time evolution of the nonlinear Fourier data. In this sense, in spite of the fact that the inverse scattering transform is applicable to nonlinear PDEs, this method still follows the logic of separation of variables.

After the emergence of a method for solving the *initial value problem* for nonlinear integrable evolution equations, the most significant outstanding open problem in the analysis of these equations became the solution of *initial-boundary value problems*. A general approach for solving such problems for evolution equations was introduced in [2] and developed in the work of more than 70 researchers [3]. It is remarkable that these results have motivated the discovery of a new transform method for solving *linear* PDEs in two variables. This method, which is referred to in the current literature as the *unified* *transform*, is based on two novel ideas (steps): (i) *perform the simultaneous spectral analysis of both equations defining the Lax pair of the given PDE—or equivalently of a certain closed differential one-form* (this is to be contrasted with the case of initial value problems, where the spectral analysis of only the t-independent part of the Lax pair is performed), and (ii) *analyze a certain global relation between all initial and boundary values*. The unified transform goes *beyond* separation of variables. Indeed, since it is based on the *simultaneous* spectral analysis of both parts of the Lax pair, it corresponds to the *synthesis* as opposed to separation of variables. As a consequence of this fundamental difference, even in the case of linear PDEs the form of the solution obtained by the unified transform differs drastically from the classical representations. It should be noted that the integral representations obtained via Green's functions retain global features. Actually, it was shown in [4, 5] that in the case of linear PDEs an alternative way to construct the novel integral representations obtained by the unified transform is to use appropriate contour deformations and Cauchy's theorem starting from the integral representations obtained via Green's functions (instead of performing the simultaneous spectral analysis of the associated Lax pair). In this sense, the unified transform reveals a deep relationship between the seminal contributions of Fourier, Cauchy, and Green and furthermore extends these contributions to integrable *nonlinear* PDEs. Indeed, it is shown in [6] that for linear PDEs this method provides a unification as well as a significant extension of the classical transforms, of the method of images, of the Green's function representations and of the Wiener–Hopf technique. (The latter technique through a series of ingenious steps gives rise to a Wiener–Hopf factorization problem, which is actually equivalent to a Riemann–Hilbert (RH) problem; in the new method, such RH problems can be *immediately* obtained using the global relation.) Furthermore, the new approach provides an appropriate "nonlinearization" of some of the above concepts.

For the case of linear elliptic PDEs, a case of great significance in many different areas of application, one of the most powerful numerical approaches is the so-called *boundary element method*. This method, which is well developed and is an established area of expertise within the numerical analysis community, can be viewed as the counterpart of the numerical implementation of the unified transform for linear elliptic PDEs. In particular, both methods are based on variational formulations. For second-order linear elliptic PDEs, such variational formulations are derived after multiplying by a test function and integrating by parts or, equivalently, are based on Green's identities. However, while the boundary element method is formulated in the physical space, the unified transform is formulated in the spectral (Fourier) space.

This book summarizes some recent developments and applications of the unified transform method and presents modern applications of the boundary element method. The first part of the book is devoted to recent advances in the applications of the unified transform, for linear and integrable nonlinear PDEs of both evolution and elliptic types. The second part consists of an important chapter by E. A. Spence that explicitly sets the stage for variational formulations of both the boundary element and unified transform methods. The last part of the book presents numerical strategies for elliptic PDEs, based on either the unified transform or the boundary element method, or the numerical implementation of the unified transform for evolution PDEs.

1.1.1 ▪ Evolution PDEs in One and Two Spatial Dimensions

Linear PDEs with *second*-order spatial derivatives formulated in one spatial dimension, with Dirichlet, Neumann, or Robin boundary conditions, can be solved by employing an appropriate transform pair in the spatial variable. For example, the heat equation on the half-line, with either Dirichlet or Neumann boundary conditions, can be solved by either the sine or the cosine transform.

Indeed, the Dirichlet problem for the heat equation on the half-line

$$u_t = u_{xx}, \quad 0 < x < \infty, \quad 0 < t < T, \quad T > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad 0 < x < \infty, \tag{1.2}$$

$$u(0,t) = f_0(t), \quad 0 < t < T, \tag{1.3}$$

where $u_0(x)$ and $f_0(t)$ have appropriate decay and smoothness, can be solved by the sine transform:

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \mathrm{e}^{-\lambda^2 t}\sin(\lambda x)\left[\int_0^\infty u_0(\xi)\sin(\lambda\xi)d\xi + \lambda\int_0^t \mathrm{e}^{\lambda^2 s} f_0(s)ds\right]d\lambda. \tag{1.4}$$

However, the λ-integral on the right-hand side of the above expression is *not* uniformly convergent with respect to the parameter x. This lack of uniformity, in addition to rendering such expressions problematic for numerical computations, also makes it difficult to verify that the solution to an initial-boundary value problem obtained using such a transform is indeed a solution. In this respect we note that the construction of the solution via any transform method is usually carried out *assuming* that a solution exists. Thus, unless one can appeal to PDE existence results, one must verify that the final formula obtained in this way does indeed satisfy the given PDE and the prescribed initial and boundary conditions. For initial-boundary value problems, this is straightforward, but for initial-boundary value problems one must overcome the lack of uniform convergence in this representation of the solution. It is interesting to note that this problem is not addressed in any of the standard applied texts on boundary value problems for linear PDEs.

It should be emphasized that the problem of lack of uniform convergence occurs in *every* nonhomogeneous boundary value problem solved by standard transform methods.

The situation is even less satisfactory for linear PDEs involving third-order spatial derivatives. A typical example is the Stokes equation on the half-line:

$$u_t + u_x + u_{xxx} = 0, \quad 0 < x < \infty, \quad 0 < t < T. \tag{1.5}$$

It can be shown that this problem, supplemented with the initial and boundary conditions (1.2) and (1.3), is well-posed. However, there does not exist an analogue of the sine or cosine transforms. One may attempt to solve the above problem by a Laplace transform in t, but this approach has several drawbacks. In particular, if $\tilde{u}(x,s)$ denotes the Laplace transform of $u(x,t)$, then $\tilde{u}$ satisfies

$$\tilde{u}_{xxx}(x,s) + \tilde{u}_x(x,s) + s\tilde{u}(x,s) = u_0(x), \quad 0 < x < \infty. \tag{1.6}$$

Thus, in order to construct an appropriate Green's function for this ODE, one must solve the homogeneous version of (1.6), that is, the cubic equation

$$\lambda(s)^3 + \lambda(s) + s = 0.$$

In contrast to the above difficulties, the unified transform (a) can be applied to linear evolution PDEs involving spatial derivatives of any order and (b) yields expressions which are always uniformly convergent at the boundary.

In particular, for the Dirichlet problem for the heat equation on the half-line, instead of (1.4), the unified transform method yields

$$u(x,t) = \frac{1}{2\pi}\left\{\int_{-\infty}^\infty \mathrm{e}^{i\lambda x - \lambda^2 t}\hat{u}_0(\lambda)d\lambda - \int_{\partial D^+} \mathrm{e}^{i\lambda x-\lambda^2 t}[\hat{u}_0(-\lambda) + 2i\lambda\tilde{f}_0(\lambda^2)]d\lambda\right\}, \tag{1.7}$$

where ∂D^+ is the boundary of the region

$$D^+ = \{\lambda \in \mathbb{C} : \mathrm{Im}\lambda \geq 0, \mathrm{Re}(\lambda^2) \leq 0\},$$

$\hat{u}_0(\lambda)$ is the Fourier transform of the initial condition, and $\tilde{f}_0(\lambda)$ is the following transform of the boundary condition:

$$\tilde{f}_0(\lambda) = \int_0^T e^{\lambda s} f_0(s) ds.$$

In contrast to (1.4), it is straightforward to verify that the function defined by (1.7) satisfies the boundary condition at $x = 0$.

Using Cauchy's theorem, the contours in (1.7) can be deformed to obtain (1.4). However, in general it is *not* possible to deform the contours in the expression obtained by the new method to obtain an expression involving integrals on the real axis. For example, such a deformation does *not* exist for the analogous expression for equation (1.5), which is consistent with the fact that in this case there does not exist a classical x-transform. Furthermore, even when such a deformation is possible, it appears that the expression formulated in the complex λ-plane has both analytical and numerical advantages. Regarding the latter advantage, we note that at least for those cases that the Fourier transform of the initial and boundary data can be computed explicitly, the computation of $u(x,t)$ reduces simply to the computation of an integral in the complex λ-plane. This integral contains the term $\exp(i\lambda x)$, which decays exponentially for large λ in the upper-half λ plane. In addition, by an appropriate contour deformation it is possible to map ∂D^+ to a contour in $\mathbb{C}^+$ with the property that that term involving the temporal dependence $\exp(\omega(\lambda)t)$ also decays exponentially for large t. Thus, the numerical computation of this integral is most efficient; see [7]. The implementation of the unified transform for linear evolution PDEs in one and two dimensions was recently reviewed in [8].

In Section 2.1, Mantzavinos uses the heat equation in one and two dimensions as an illustrative example to show that, using the unified transform, it is possible to construct analytic solutions for problems involving nonseparable boundary conditions as well as nonlocal constraints. It should be noted that even for the particular cases that the boundary conditions are separable, the unified transform gives a construction of the relevant spectral representations in a much simpler way than the classical spectral analysis. Actually, the unified transform provides a new approach to some open problems in spectral analysis. In Section 2.2, Smith illustrates the spectral interpretation of the unified transform, describing how this method can be viewed as the natural extension of Fourier transform techniques for non-self-adjoint operators. This section also discusses the spectral meaning of the transform pair defined by the unified transform, using the recent definition of a new class of spectral functionals that essentially diagonalize the non-self-adjoint spatial differential operators associated with linear evolution boundary value problems.

A great advantage of the unified transform for linear evolution initial-boundary value problems is that it provides an algebraic way for eliminating the unknown boundary values or, more precisely, appropriate transforms of the unknown boundary values. However, for integrable nonlinear PDEs this elimination is in general *not* possible. Thus, for such nonlinear PDEs one has to characterize the so-called *generalized Dirichlet-to-Neumann map*, namely characterize the unknown boundary values in terms of the given initial and boundary data. However, it turns out that for a particular class of boundary conditions, called *linearizable*, the algebraic elimination *is* possible, and hence for the corresponding problems the unified transform method is as effective as the classical inverse scattering transform. Recall that the sine and cosine transforms can be obtained from the Fourier transform by considering either an odd or an even symmetry. Thus, if an initial boundary value problem can be solved by a sine or cosine transform, it can also be solved by an odd or even extension to the full line via the Fourier transform. The situation is similar for linearizable integrable PDEs posed on the half-line. For example, linearizable initial boundary value problems for the nonlinear Schrödinger (NLS) equation (which is a second-order PDE whose linear version can therefore be solved by an appropriate sine or cosine transform) can be solved either by the unified transform or by the inverse scattering transform after an appropriate extension to the full line. However, linearizable initial-boundary value problems for the Korteweg–deVries (KdV) equation (whose linear version *cannot* be solved by an appropriate x-transform) so far can only be solved by the unified transform. These issues are further discussed in Chapter 3, which discusses in detail the relation between the two above approaches to the solution of initial-boundary value problems for the NLS equation on the half-line with linearizable boundary conditions.

1.1.2 ▪ Elliptic PDEs

It is well known that the main difficulty with boundary value problems stems from the fact that, although the solution representation requires the knowledge of all boundary values, some of them are *not* prescribed as boundary conditions. In the theory of elliptic PDEs, the determination of the unknown boundary values is known as the problem of characterizing the Dirichlet-to-Neumann map. The global relation provides the starting point for analytical, rigorous, and numerical characterizations of this map. Regarding analytical results, we note that for certain simple domains, the unified transform yields analytical expressions for the unknown boundary values [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20]. Regarding rigorous results, we note that in a series of important papers, Ashton has shown that the analysis of the global relation provides a novel approach to obtaining a plethora of rigorous results for linear elliptic PDEs. This rigorous approach is summarized in Chapter 4. Regarding numerical results, we note that the unified transform has inspired a novel numerical technique for the determination of the unknown boundary values [21], [22], [23], [24], [25], [26], [27], [28], [29], [30]. For elliptic PDEs formulated in the interior of a polygon, the above technique provides the analogue of the boundary integral method, but now the analysis takes place in the spectral (Fourier) space instead of the physical space. This technique is summarized in Section 7.1 by Fokas, Iserles, and Smitheman. Furthermore, the first steps toward extending this technique to the *exterior* of a polygon are presented in Section 7.2, by Fokas and Lenells.

Part II of the book, by Spence, presents an overview of variational formulations of second-order linear elliptic PDEs that are based on integration by parts or, equivalently, on Green's identities.

The difficulty of treating effectively boundary value problems in the exterior of a bounded domain is at the heart of Section 7.3, where Chandler-Wilde and Langdon discuss the applications of the boundary element method to high-frequency problems arising in acoustics and make connections to the unified transform by analyzing particular instances of this method for problems of acoustic scattering by diffraction gratings. The practical problem is the scattering of acoustic waves by an obstacle is naturally formulated as a boundary value problem in the domain exterior to the obstacle. The efficient solution of such problems, in particular the resolution of the high-frequency components, is still a difficult problem, and what is discussed is a state-of-the-art hybrid boundary element method for it, as well as connections to the unified transform.

The same difficulty is again the main motivation for Section 7.4, where Claeys et al. consider scalar second-order problems in the exterior of a bounded domain with Dirichlet boundary conditions on the boundary and formulate two novel well-posed *multi-trace* *boundary integral equations* for such problems. Compared to conventional single-trace formulations they offer the advantage of being amenable to operator preconditioning, which allows greater numerical effectiveness.

The first applications of the unified transform to nonlinear elliptic PDEs were presented in [31, 32]. In Section 5.1 the results to date on the case of the elliptic sine-Gordon equations are summarized by Pelloni, who also gives a complete characterization of linearizable boundary conditions for the case of boundary value problems posed in a semi-infinite strip.

A crucial step in the development of the unified transform was the observation by Fokas and Gelfand that linear PDEs always admit a Lax pair formulation [33]. This observation also played a crucial role in the development of a new technique for deriving transforms, which finally led to the construction by Novikov of the inverse attenuated Radon transform [34]. It is remarkable that as shown by Trogdon, this formulation is also useful for the numerical integration of the initial value problem of linear evolution PDEs. In addition to this result, Trogdon presents in section 8.1 a powerful new technique for the numerical solution of integrable PDEs based on the numerical integration of the associated RH formulation.

Conclusions and Open Problems

Following its introduction in [2], the unified transform has been extensively developed, and it has found applications in surprisingly many and varied mathematical fields. This book touches upon recent developments and applications.

Several important problems remain open, including the following:

(a) *t-periodic boundary conditions.* It was noted earlier that initial-boundary value problems for evolution PDEs on the half-line with *linearizable* boundary conditions can be solved via the unified transform as efficiently as initial value problems. Furthermore, for *nonlinearizable* boundary conditions which decay for large t, the unified transform yields useful asymptotic information on the large t behavior of the solution. However, for the physically important case of *t-periodic* boundary conditions, it is necessary to characterize the Dirichlet-to-Neumann map. Pioneering results in this direction for the NLS have been obtained in [35, 36, 37]. However, several problems, including the extension of these results for the case of KdV, remain open.

(b) *x-periodic initial conditions.* The initial value problem for nonlinear integrable PDEs with *periodic* initial conditions can be analyzed via the elegant algebro-geometric formalism. This involves finite-genus Riemann surfaces and yields finite-dimensional solutions sets [38, 39, 40]. Even if these solutions could be made computationally accessible [41, 42], one still has to answer the important question of the density of these solutions in an appropriate function space. It has been shown in [43] that this problem belongs to the linearizable class. However, the relevant implementation of the unified transform in this case remains open.

(c) *Three-dimensional problems.* The implementation of the unified transform to linear evolution PDEs in two spatial dimensions is discussed in the article of Mantzavinos. Furthermore, the first steps toward the analysis of integrable nonlinear evolution PDEs in two spatial dimensions have been taken in [44]. However, the associated analysis of the Dirichlet-to-Neumann map, as well as the implementation of the unified transform to elliptic PDEs in three dimensions, remain open.

The approach described and applied in the various sections contained in this book need to be developed further. The articles presented here, together with the extensive