# Vectors and Matrices

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# *Vectors and Matrices*

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# Vectors and Matrices

to JANETTE, DAVID, and PAUL

# *Editors' Preface*

This series is a new approach to the ever growing problem of providing appropriate material for basic mathematics courses offered for those planning careers in fields other than mathematics. Rapidly changing concepts in what should be offered at the undergraduate level have led to many approaches with various arrangements of topics. A serious problem arises in an effort to implement all of the material needed to satisfy the needs of these different courses. Too often there is no single book that treats the material in the desired manner or includes the desired topics at an acceptable level.

This series is planned as an integrated group of high quality books, each complete within itself except for required background material, each covering a specified topic, and each preparing the reader for the topics that would follow naturally.

With the flexibility that such a series offers, it is hoped that every requirement can be satisfied by a careful selection of books. In designing the series, we have tried to meet two requirements. We have tried, first, to satisfy the need for flexibility in the subject material; and secondly, to present to the reader material that has been prepared by an author with a specialized background in that particular area.

In editing this series, we have insisted that each author treat his subject material in a way to give it operational meaning. With the specialist's greater familiarity in a particular subject, he can delicately merge the abstract with the practical. He can give a functional interpretation to the concepts of mathematics, thereby motivating the student and creating the necessary interest to make the learning experience exciting.

We owe a deep debt of gratitude to each author who has contributed to the series. Further, we are grateful to Charles E. Merrill

Publishing Company, for the assistance they have given us in the development of this series.

Austin, Texas
Pittsburgh, Pennsylvania

Vincent E. Cangelosi
Melvin J. Hinich

# Author's Preface

In recent years, quantitative methods, especially those that utilize concepts of linear algebra, have been applied to a wide array of problems in management. This book will provide an introduction to topics from the theory of vectors and matrices that are most useful in formulating and solving problems related to management.

*Vectors and Matrices* has a more narrative approach than is usually found in books on the topic. The book presents basic concepts and theorems in the form of narrative text, followed generally by demonstrations and proofs in the form of examples and illustrations. As far as possible, the mathematical ideas of vectors and matrices are presented first in words and then in symbolic form. Important concepts, statements, and corresponding results are set off in italic type.

The text consists of four principal sections. The first treats concepts of vectors including vector algebra, operations, geometric representations, and applications of vectors to linear equations. The second section presents determinants and techniques of finding the numerical value of an array of numbers. The third section includes concepts of matrix algebra, operations, properties of matrices, selected advanced topics, and applications of matrices to systems of linear equations. The last section concerns applications of vectors and matrices in management. The concepts in the sections are, for the most part, elementary, and have been selected for their usefulness in management. A more exhaustive treatment of topics that infrequently have operational meaning in management is beyond the scope of this presentation.

The author is indebted to many persons for their encouragement and assistance in the development of this manuscript. In particular, appreciation is expressed to the series' editors, Professor Vincent E.

Cangelosi of The University of Texas and Melvin J. Hinich of Carnegie Institute of Technology, for their very helpful comments and suggestions. Special recognition is given to the Graduate School of Management, Instituto Tecnologico de Monterrey, Mexico, for providing an atmosphere conducive to writing the manuscript. The author is indebted to Janette K. Schkade for her invaluable moral support and for her assistance in editing and typing the manuscript.

Lawrence L. Schkade

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## chapter one

# *The Algebra of Vectors*

Many problems in business involve groups or collections of numbers that are ordered or arrayed into columns or rows. Arithmetic operations are applied to these ordered collections of numbers to obtain problem solutions. For example, arithmetic operations are applied to solve simultaneous equations, and in the process, rows or ordered numbers are multiplied or added. This chapter presents the basic concepts of vectors and vector operations.

## 1–1. BASIC TYPES OF VECTORS

Several different types of vectors are used commonly in the formulation and solution of business problems. These types are defined in the following sections.

## Column Vectors

*An ordered group or collection of numbers arranged to form a column is called a column vector.* These numbers are enclosed in brackets to designate that the numbers comprise a specific vector.

*Example*

Some groups of numbers arranged to form column vectors are

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0.1 \\ 0.6 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1.0 \\ -2.3 \\ 0.0 \end{bmatrix}.$$

Each of the numbers in a vector is called an *element* or *component*. In this presentation, an element is assumed to be any one of the set of real numbers. These include all positive or negative integers or fractions. It is assumed that none of the elements is a complex number.

The number of elements in a vector is a principal characteristic of a vector. In the foregoing examples, some vectors have two elements and are called two-element vectors. Other vectors with three elements are called three-element vectors. The general form of a column vector is referred to as an *m-element* column vector or an $m \times 1$ vector and is denoted

$$\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}.$$

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## Row Vector

A group of numbers may also be ordered to form a row. *An array of this sort is called a row vector.*

*Example*

Some groups of numbers arranged to form row vectors are

$$[0 \quad 1], \quad [-3 \quad 2], \quad [0.2 \quad 0.1 \quad -0.7], \quad [-8.1 \quad 6.2 \quad 0.5 \quad -4.9].$$

Each of the numbers in a row vector is called an element and is an integer, a rational number, or a real number. The number of components in a row vector, as in the case of a column vector, is a principal distinguishing characteristic. In general form, a row vector is denoted as *n-element* vector or a $1 \times n$ vector written

$$\mathbf{V} = [v_1 \quad v_2 \cdots v_n].$$

## Zero Vector

*A special type of vector for which all elements have a value of zero is called a zero vector.*

## Equal Vectors

*Any two row vectors, or two column vectors, that have elements that are equal and in the same order are called equal vectors.*

*Example*

Consider the $1 \times 3$ and $3 \times 1$ vectors

$$\mathbf{R} = [3 \quad 1 \quad 0], \quad \mathbf{S} = [1 \quad 0 \quad 3], \quad \mathbf{T} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{U} = [3 \quad 1 \quad 0].$$

Vectors $\mathbf{R}$ and $\mathbf{U}$ are equal since the corresponding elements are the same. Similarly, $\mathbf{R} \neq \mathbf{S}$ since the corresponding elements are not equal. The vectors $\mathbf{S}$ and $\mathbf{T}$ are not equal because $\mathbf{S}$ is a row vector and $\mathbf{T}$ is a column vector.

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## Unit Vector

*A vector that has one component with a value of one and all other components equal to zero is called a unit vector.* Some examples are

$$\mathbf{U} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

## Transpose of a Vector

Suppose a $3 \times 1$ vector written

$$\mathbf{U} = \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$$

is rewritten in horizontal form such that the order of the elements is preserved. The result is written

$$\mathbf{V} = [3 \quad 6 \quad 4].$$

*This* $1 \times 3$ *vector is called the tranpose of the* $3 \times 1$ *column vector* **U**. Similarly, the transpose of the row vector **V**, denoted $\mathbf{V}^T$, is the column vector **U**. Thus,

$$\mathbf{U}^T = \mathbf{V}, \quad \mathbf{V}^T = \mathbf{U}, \text{ and } [\mathbf{V}^T]^T = \mathbf{V}.$$

## 1–2. VECTOR OPERATIONS

Vectors are convenient mathematical devices with which to express ordered groups of values, especially if the number of elements is large. Usually, the solution of a problem involving vectors requires that ordered groups of values be added, subtracted, or multiplied. These basic arithmetic operations may be applied to vectors according to rules that are analogous to those of ordinary algebra. The concept of division is not analogous for vectors and matrices in the same sense as the other operations, and the discussion of this operation is best presented in the section on matrix operations.

### Addition

*The addition of vectors requires that the vectors to be added have the same dimensions.* It follows that addition involving a $2 \times 1$ column vector is defined only for column vectors of the same dimensionality. In the addition of vectors, corresponding elements are added. Consider the vectors

$$\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

The addition of these vectors is accomplished by adding corresponding elements. In this manner the additions are $u_1 + v_1$, $u_2 + v_2$, and $u_3 + v_3$. This addition is written

$$\mathbf{U} + \mathbf{V} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}.$$

*Example*

The following vectors are to be added:

$$\mathbf{W} = \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix}.$$

The addition of the $3 \times 1$ vectors is accomplished thus:

$$\mathbf{W} + \mathbf{Y} + \mathbf{Z} = \begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 4+0-2 \\ 6-1+0 \\ 2+3-4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}.$$

---

The addition of row vectors is performed in a manner similar to that in the addition of column vectors.

*Example*

Three $1 \times 2$ vectors are to be added. These vectors are written

$$\mathbf{R} = [1 \quad 0], \quad \mathbf{S} = [-3 \quad 2], \quad \mathbf{T} = [6 \quad -4].$$

The addition of the vectors is accomplished by writing

$$\begin{aligned} \mathbf{R} + \mathbf{S} + \mathbf{T} &= [1 \quad 0] + [-3 \quad 2] + [6 \quad -4] = [1 - 3 + 6][0 + 2 - 4] \\ &= [4 \quad -2]. \end{aligned}$$

---

Using the foregoing example, it is easy to see that

$$\mathbf{R} + \mathbf{S} + \mathbf{T} = \mathbf{R} + \mathbf{T} + \mathbf{S} = \mathbf{S} + \mathbf{T} + \mathbf{R} = \ldots = \mathbf{T} + \mathbf{S} + \mathbf{R}.$$

In words, whatever the order in which the vectors are added, the sum is the same, so long as the vectors are of the same type and dimensionality. *This demonstrates that the Commutative Law of Addition applies to vector addition.*

Another important characteristic of vector addition is illustrated by use of the vectors from the foregoing example. Changing the addition procedure slightly, then

$$\mathbf{R} + (\mathbf{S} + \mathbf{T}) = (\mathbf{R} + \mathbf{S}) + \mathbf{T}$$
$$[1 \quad 0] + ([-3 \quad 2] + [6 \quad -4]) = ([1 \quad 0] + [-3 \quad 2]) + [6 \quad -4]$$
$$[4 \quad -2] = [4 \quad -2].$$

*This equality demonstrates that the Associative Law of Addition applies to vector addition.*

## Subtraction

The subtraction of vectors is a special case of the rules for addition. *Subtraction is achieved by taking the negative of the vector that is the subtrahend and adding the vectors algebraically.* The subtraction of the vector $\mathbf{V}$ from the vector $\mathbf{U}$ is written

$$\mathbf{U} - \mathbf{V} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} -v_1 \\ -v_2 \\ -v_3 \end{bmatrix} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{bmatrix}.$$

*Example*

A $3 \times 1$ vector $\mathbf{Y}$ is to be subtracted from the vector $\mathbf{X}$. These vectors are written

$$\mathbf{X} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.$$

The subtraction of these vectors is achieved by the procedure written

$$\mathbf{X} - \mathbf{Y} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

---

The subtraction of row vectors is accomplished in an analogous manner.

*Example*

The row vector $\mathbf{B}$ is to be subtracted from the vector $\mathbf{A}$. These vectors are written

$$\mathbf{A} = [6 \quad -3 \quad 4 \quad 7], \quad \mathbf{B} = [3 \quad 5 \quad 0 \quad 8].$$

The subtraction is given by

$$\begin{aligned} \mathbf{A} - \mathbf{B} &= [6 \quad -3 \quad 4 \quad 7] - [3 \quad 5 \quad 0 \quad 8] \\ &= [6-3 \quad -3-5 \quad 4-0 \quad 7-8] \\ &= [3 \quad -8 \quad 4 \quad -1]. \end{aligned}$$

---

As in the addition of vectors, subtraction is defined only for vectors of the same type and dimensionality.

## Scalar Multiplication

The rules for vector multiplication build on those defined for vector addition. For example, if two equal vectors are added, the result is a vector with elements that have twice the value as the corresponding elements in either of the two vectors. This relationship is illustrated thus:

$$\mathbf{U} + \mathbf{V} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3+3 \\ -1-1 \\ 2+2 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix}.$$

*The result of the addition* $\mathbf{U} + \mathbf{V}$ *may be viewed as the product of a number or scalar and a vector*. The appropriate multiplier in the foregoing example is two, and