



Frontiers in Complex Dynamics

In Celebration of John Milnor's 80th Birthday

EDITED BY ARACELI BONIFANT,
MISHA LYUBICH,
& SCOTT SUTHERLAND

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Frontiers in Complex Dynamics

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Preface

In February of 2011, over 130 mathematicians gathered at the beautiful Banff Centre in the Canadian Rockies for a week of discussing holomorphic dynamics in one and several variables and other topics related to the work of John Milnor.

John Milnor is undoubtedly one of the most significant mathematicians of the second half of the twentieth century. He has made fundamental discoveries in many areas of modern mathematics, including topology, geometry, K-theory, and dynamical systems. Since in recent years his main interest has been in complex dynamics, it was only fitting that the conference had this as a primary focus.

The conference in Banff was a great success. In addition to all of the wonderful mathematics, the beautiful setting and friendly atmosphere at the Banff Centre inspired us all, both professionally and personally. All but one of the talks were videotaped and can be viewed or downloaded from the conference website, at <http://www.math.sunysb.edu/jackfest> (unfortunately, there was a camera malfunction at the start of Arnaud Chéritat's presentation, so only his slides are available).

This collection is an outgrowth of that conference, which was also organized by the editors of this book. Almost all of the authors whose papers appear here attended the conference. Both this collection and the conference were designed to honor John Milnor. But this volume is not merely a record of that conference; rather, it extends and complements that event. For example, some of the speakers gave primarily expository lectures but chose to contribute research papers to this volume; others went the other route. There is very little skiing in the book, and you'll have to bring your own food. But, it should last longer. We hope this volume will be valuable to any mathematician working in complex dynamics or related fields, whether or not they attended the Banff conference.



This volume is organized in five main parts: *I. One Complex Variable*, *II. One Real Variable*, *III. Several Complex Variables*, *IV. Laminations and Foliations*, and *V. Geometry and Algebra*. The first part is further subdivided into the areas of *arithmetic dynamics*, *polynomial dynamics*, *rational dynamics*, and *Thurston theory*, and the third part first covers *local dynamics in several complex variables* and then turns to *global dynamics*. In addition, there is a section containing color versions of those images for which color is essential; such images have references within the body of the main text, where a greyscale version appears for the reader's convenience.

The photograph of Jack Milnor at Lake Louise (Plate 1) was taken by Thomas Milnor. The images in Figure 1.2 on page 75 (which appears in modified form as Plate 4) are reprinted from John Milnor's article "Remarks on Iterated Cubic

Maps" in *Experimental Mathematics* 1, no. 1 (1992) by permission of Taylor & Francis (<http://www.tandfonline.com>). The image in Plate 15 (which also appears in modified form as Figure 3.11 on page 151) was produced by Hiroyuki Inou. The images in Plate 23 and Figure 1.1 on page 465 were produced by Vincent Pit. The group photo of conference participants (Plates 29 and 30) was taken by Photographic Services, The Banff Centre. The conference poster on page C-24 used elements from photographs taken by Marco Martens and by Tom Arban Photography (<http://www.tomarban.com>). All images are used by permission.



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Contents

Preface	xi
Introduction	1
1. Bibliography	8
Part I. One Complex Variable	13
J. MILNOR: Arithmetic of Unicritical Polynomial Maps	15
1. Introduction	15
2. Periodic orbits	15
3. Proofs	18
4. Postcritically finite maps	20
5. Bibliography	23
T. BOUSCH: Les racines des composantes hyperboliques de M sont des quarts d'entiers algébriques	25
A. BLOKH, L. OVERSTEEGEN, R. PTACEK, AND V. TIMORIN: Dynamical cores of topological polynomials	27
1. Introduction and the main result	27
2. Preliminaries	29
3. Dynamical core	35
4. Bibliography	47
X. BUFF AND TAN LEI: The quadratic dynatomic curves are smooth and irreducible	49
1. Introduction	49
2. Dynatomic polynomials	50
3. Smoothness of the dynatomic curves	53
4. Irreducibility of the dynatomic curves	61
5. Bibliography	71
J. H. HUBBARD AND D. SCHLEICHER: Multicorns are not path connected	73
1. Introduction	73
2. Antiholomorphic and parabolic dynamics	77
3. Bifurcation along arcs and the fixed-point index	80
4. Parabolic perturbations	85
5. Parabolic trees and combinatorics	89
6. Non-pathwise connectivity	95
7. Further results	98
8. Bibliography	101

J. KIWI: Leading monomials of escape regions	103
1. Introduction	103
2. Statement of the results	104
3. Puiseux series dynamics statement	106
4. One-parameter families	108
5. Bibliography	118
R. L. DEVANEY: Limiting behavior of Julia sets of singularly perturbed rational maps	121
1. Introduction	121
2. Elementary mapping properties	123
3. Julia sets converging to the unit disk	126
4. The case $n > 2$	128
5. Other c -values	131
6. Bibliography	133
A. DEZOTTI AND P. ROESCH: On (non-)local connectivity of some Julia sets	135
1. Local connectivity	136
2. Rational maps	141
3. Douady-Sullivan criterion	143
4. The case of infinitely satellite-renormalizable quadratic polynomials: a model	150
5. Bibliography	161
G. LEVIN: Perturbations of weakly expanding critical orbits	163
1. Introduction	163
2. Polynomials	167
3. Rational functions	176
4. Part (a) of Theorem 3.6	182
5. Part (b) of Theorem 3.6	189
6. Bibliography	195
D. MEYER: Unmating of rational maps: Sufficient criteria and examples	197
1. Introduction	197
2. Moore's theorem	199
3. Mating of polynomials	201
4. Equators and hyperbolic rational maps	204
5. An example	209
6. A sufficient criterion for mating	211
7. Connections	216
8. Critical portraits	220
9. Unmating the map	221
10. Examples of unmatings	224
11. A mating not arising from a pseudo-equator	228
12. Open questions	232
13. Bibliography	232

Y. JIANG: A framework toward understanding the characterization of holomorphic dynamics	235
1. Characterization	235
2. Obstruction	237
3. Review	237
4. Geometry	242
5. Geometrization	248
6. Appendix on Transcendental Functions by Tao Chen, Yunping Jiang, and Linda Keen	252
7. Bibliography	254
Part II. One Real Variable	259
C. G. MOREIRA AND D. SMANIA: Metric stability for random walks (with applications in renormalization theory)	261
1. Introduction	261
2. Expanding Markov maps, random walks, and their perturbations	265
3. Statements of results	269
4. Preliminaries	272
5. Stability of transience	276
6. Stability of recurrence	288
7. Stability of the multifractal spectrum	296
8. Applications to one-dimensional renormalization theory	312
9. Bibliography	321
S. VAN STRIEN: Milnor's conjecture on monotonicity of topological entropy: Results and questions	323
1. Motivation	323
2. Milnor's monotonicity of entropy conjecture	324
3. Idea of the proof	326
4. Open problems	331
5. Bibliography	334
W. P. THURSTON: Entropy in dimension one	339
1. Introduction	339
2. Special case: Pisot numbers	346
3. Constructing interval maps: First steps	351
4. Second step: Constructing a map for λ^N	355
5. Powers and roots: Completion of proof of Theorem 1.3	355
6. Maps of asterisks	358
7. Entropy in bounded degree	359
8. Traintracks	365
9. Splitting hairs	367
10. Dynamic extensions	369
11. Bipositive matrices	370
12. Tracks, doubletracks, zipping and a sketch of the proof of Theorem 1.11	374
13. Supplementary notes (mostly by John Milnor)	376
14. Bibliography	383

Part III. Several Complex Variables	385
M. ARIZZI AND J. RAISSY: On Écalte-Hakim's theorems in holomorphic dynamics	387
1. Introduction	387
2. Notation	389
3. Preliminaries	389
4. Characteristic directions	390
5. Changes of coordinates	397
6. Existence of parabolic curves	407
7. Existence of attracting domains	421
8. Parabolic manifolds	423
9. Fatou coordinates	439
10. Fatou-Bieberbach domains	444
11. Bibliography	447
M. ABATE: Index theorems for meromorphic self-maps of the projective space	451
1. Introduction	451
2. The proof	453
3. Bibliography	460
S. CANTAT: Dynamics of automorphisms of compact complex surfaces	463
1. Introduction	463
2. Hodge theory and automorphisms	466
3. Groups of automorphisms	476
4. Periodic curves, periodic points, and topological entropy	478
5. Invariant currents	485
6. Entire curves, stable manifolds, and laminarity	489
7. Fatou and Julia sets	498
8. The measure of maximal entropy and periodic points	502
9. Complements	505
10. Appendix: Classification of surfaces	507
11. Bibliography	509
R. DUJARDIN: Bifurcation currents and equidistribution in parameter space	515
1. Prologue: normal families, currents and equidistribution	518
2. Bifurcation currents for families of rational mappings on \mathbb{P}^1	524
3. Higher bifurcation currents and the bifurcation measure	534
4. Bifurcation currents for families of Möbius subgroups	547
5. Further settings, final remarks	557
6. Bibliography	559
Part IV. Laminations and Foliations	567
T.-C. DINH, V.-A. NGUYÊN AND N. SIBONY: Entropy for hyperbolic Riemann surface laminations I	569
1. Introduction	569

2. Poincaré metric on laminations	571
3. Hyperbolic entropy for foliations	578
4. Entropy of harmonic measures	585
5. Bibliography	591
T.-C. DINH, V.-A. NGUYÊN AND N. SIBONY: Entropy for hyperbolic Riemann surface laminations II	593
1. Introduction	593
2. Local models for singular points	595
3. Poincaré metric on leaves	604
4. Finiteness of entropy: the strategy	611
5. Adapted transversals and their coverings	614
6. Finiteness of entropy: end of the proof	619
7. Bibliography	621
V. MUÑOZ AND R. PÉREZ-MARCO: Intersection theory for ergodic solenoids	623
1. Introduction	623
2. Measured solenoids and generalized currents	624
3. Homotopy of solenoids	627
4. Intersection theory of solenoids	630
5. Almost everywhere transversality	638
6. Intersection of analytic solenoids	641
7. Bibliography	644
H. GARCÍA-COMPEÁN, R. SANTOS-SILVA AND A. VERJOVSKY: Invariants of four-manifolds with flows via cohomological field theory	645
1. Introduction	645
2. Asymptotic cycles and currents	647
3. Overview of cohomological quantum field theory: Donaldson-Witten invariants	650
4. Donaldson-Witten invariants for flows	653
5. Donaldson-Witten invariants for Kähler manifolds with flows	659
6. Survey on Seiberg-Witten invariants	667
7. Seiberg-Witten invariants for flows	669
8. A physical interpretation	672
9. Final remarks	673
10. Bibliography	674
Color Plates	C-1
Part V. Geometry and Algebra	677
W. GOLDMAN: Two papers which changed my life: Milnor's seminal work on flat manifolds and bundles	679
1. Introduction	679
2. Gauss-Bonnet beginnings	679
3. The Milnor-Wood inequality	682
4. Maximal representations	684

5. Complete affine manifolds	687
6. Margulis spacetimes	691
7. Bibliography	699
R. GRIGORCHUK: Milnor's problem on the growth of groups and its consequences	705
1. Introduction	705
2. Acknowledgments	709
3. Preliminary facts	709
4. The problem and the conjecture of Milnor	713
5. Relations between group growth and Riemannian geometry	715
6. Results about group growth obtained before 1981	717
7. Growth and amenability	719
8. Polynomial growth	722
9. Intermediate growth: The construction	726
10. The gap conjecture	734
11. Intermediate growth: The upper and lower bounds	738
12. Asymptotic invariants of probabilistic and analytic nature and corresponding gap-type conjectures	743
13. Inverse orbit growth and examples with explicit growth	750
14. Miscellaneous	753
15. Bibliography	758
Contributors	775
Index	779

Introduction

Holomorphic dynamics is one of the earliest branches of dynamical systems which is not part of classical mechanics. As a prominent field in its own right, it was founded in the classical work of Fatou and Julia (see [Fa1, Fa2] and [J]) early in the 20th century. For some mysterious reason, it was then almost completely forgotten for 60 years. The situation changed radically in the early 1980s when the field was revived and became one of the most active and exciting branches of mathematics. John Milnor was a key figure in this revival, and his fascination with holomorphic dynamics helped to make it so prominent. Milnor's book *Dynamics in One Complex Variable* [M8], his volumes of collected papers [M10, M11], and the surveys [L1, L5] are exemplary introductions into the richness and variety of Milnor's work in dynamics.

Holomorphic dynamics, in the sense we will use the term here, studies iterates of holomorphic maps on complex manifolds. Classically, it focused on the dynamics of rational maps of the Riemann sphere $\widehat{\mathbb{C}}$. For such a map f , the Riemann sphere is decomposed into two invariant subsets, the *Fatou set* $\mathcal{F}(f)$, where the dynamics is quite tame, and the *Julia set* $\mathcal{J}(f)$, which often has a quite complicated fractal structure and supports chaotic dynamics.

Even in the case of quadratic polynomials $Q_c: z \mapsto z^2 + c$, the dynamical picture is extremely intricate and may depend on the parameter c in an explosive way. The corresponding bifurcation diagram in the parameter plane is called the *Mandelbrot set*; its first computer images appeared in the late 1970s, sparking an intense interest in the field [BrMa, Man].

The field of holomorphic dynamics is rich in interactions with many branches of mathematics, such as complex analysis, geometry, topology, number theory, algebraic geometry, combinatorics, and measure theory. The present book is a clear example of such interplay.



The papers "**Arithmetic of Unicritical Polynomial Maps**" and "**Les racines de composantes hyperboliques de M sont des quarts d'entiers algébriques,**" which open this volume,¹ exemplify the interaction of holomorphic dynamics with number theory. In these papers, John Milnor and Thierry Bousch study number-theoretic properties of the family of polynomials $p_c(z) = z^n + c$, whose bifurcation diagram is known as the *Multibrot set*.

In the celebrated Orsay Notes [DH1], Douady and Hubbard undertook a remarkable combinatorial investigation of the Mandelbrot set and the corresponding bifurcations of the Julia sets. In particular, they realized (using important contributions from Thurston's work [T]) that these fractal sets admit an explicit topological model as long as they are locally connected (see [D]). This led to the most famous conjecture in the field, on the local connectivity of the Mandelbrot set, typically

¹Both these papers were originally written circa 1996 but never published. Milnor's paper is a follow-up to Bousch's note, but it was significantly revised by the author for this volume.

abbreviated as *MLC*. The *MLC* conjecture is still currently open, but it has led to many important advances, some of which are reflected in this volume.

In his thesis [La], Lavaurs proved the non-local-connectivity of the cubic connectedness locus, highlighting the fact that the degree two case is special in this respect. In attempt to better understand this phenomenon, Milnor came across a curious new object that he called the *tricorn*: the connectedness locus of antiholomorphic quadratic maps $q_c(z) = \bar{z}^2 + c$. In the paper **“Multicorns are not path connected,”** John Hubbard and Dierk Schleicher take a close look at the connectedness locus of its higher degree generalization, defined by $p_c(z) = \bar{z}^n + c$.

The paper by Alexandre Dezotti and Pascale Roesch, **“On (non-)local connectivity of some Julia sets,”** surveys the problem of local connectivity of Julia sets. It collects a variety of results and conjectures on the subject, both “positive” and “negative” (as Julia sets sometimes fail to be locally connected). In particular, in this paper the reader can learn about the work of Yoccoz [H, M7], Kahn and Lyubich [KL], and Kozlovski, Shen, and van Strien [KSvS]; the latter gives a positive answer in the case of “non-renormalizable” polynomials of any degree.

Related to connectivity, an important question that has interested both complex and algebraic dynamicists is that of the irreducibility of the closure of X_n , the set of points $(c, z) \in \mathbb{C}^2$ for which z is periodic under $Q_c(z) = z^2 + c$ with minimal period n . These curves are known as *dynamotic curves*. The irreducibility of such curves was proved by Morton [Mo] using algebraic methods, by Bousch [Bou] using algebraic and analytic (dynamical) methods, and by Lau and Schleicher [LS], using only dynamical methods. In the paper **“The quadratic dynamotic curves are smooth and irreducible,”** Xavier Buff and Tan Lei present a new proof of this result based on the *transversality theory* developed by Adam Epstein [E].

Similarly, in the case of the family of cubic polynomial maps with one marked critical point, parametrized by the equation $F(z) = z^3 - 3a^2z + (2a^3 + v)$, one can study the *period p -curves* \mathcal{S}_p for $p \geq 1$. These curves are the collection of parameter pairs $(a, v) \in \mathbb{C}^2$ for which the marked critical point a has period exactly p ; Milnor proved that \mathcal{S}_p is smooth and affine for all $p > 0$ and irreducible for $p \leq 3$ [M9]. The computation of the Euler characteristic for any $p > 0$ and the irreducibility for $p = 4$ were proved by Bonifant, Kiwi and Milnor [BKM]. The computation of the Euler characteristic requires a deep study of the unbounded hyperbolic components of \mathcal{S}_p , known as *escape regions*. Important information about the limiting behavior of the periodic critical orbit as the parameter tends to infinity within an escape region is encoded in an associated *leading monomial vector*, which uniquely determines the escape region, as Jan Kiwi shows in **“Leading monomials of escape regions.”**

As we have alluded to previously, a locally connected Julia set admits a precise topological model, due to Thurston, by means of a *geodesic lamination* in the unit disk. This model can be efficiently described in terms of the *Hubbard tree*, which is the “core” that encodes the rest of the dynamics. In particular, it captures all the cut-points of the Julia set, which generate the lamination in question. This circle of ideas is described and is carried further to a more general topological setting in the paper by Alexander Blokh, Lex Oversteegen, Ross Ptacek and Vladlen Timorin **“Dynamical cores of topological polynomials.”**

The realm of general *rational dynamics* on the Riemann sphere is much less explored than that of polynomial dynamics. There is, however, a beautiful bridge connecting these two fields called *mating*: a surgery introduced by Douady and

Hubbard in the 1980s, in which the filled Julia sets of two polynomials of the same degree are dynamically related via external rays. In many cases this process produces a rational map. It is a difficult problem to decide when this surgery works and which rational maps can be obtained in this way. A recent breakthrough in this direction was achieved by Daniel Meyer, who proved that in the case when f is postcritically finite and the Julia set of f is the whole Riemann sphere, every sufficiently high iterate of the map can be realized as a mating [Me1, Me2]. In the paper **“Unmating of rational maps, sufficient criteria and examples,”** Meyer gives an overview of the current state of the art in this area of research, illustrating it with many examples. He also gives a sufficient condition for realizing rational maps as the mating of two polynomials.

Another way of producing rational maps is by “singular” perturbations of complex polynomials. In the paper **“Limiting behavior of Julia sets of singularly perturbed rational maps,”** Robert Devaney surveys dynamical properties of the families $f_{c,\lambda}(z) = z^n + c + \lambda/z^d$ for $n \geq 2$, $d \geq 1$, with c corresponding to the center of a hyperbolic component of the Multibrot set. These rational maps produce a variety of interesting Julia sets, including *Sierpinski carpets* and *Sierpinski gaskets*, as well as laminations by Jordan curves. In the current article, the author describes a curious “implosion” of the Julia sets as a polynomial $p_c = z^n + c$ is perturbed to a rational map $f_{c,\lambda}$.

There is a remarkable *phase-parameter relation* between the dynamical and parameter planes in holomorphic families of rational maps. It first appeared in the early 1980s in the context of quadratic dynamics in the Orsay notes [DH1] and has become a very fruitful philosophy ever since. In the paper **“Perturbations of weakly expanding critical orbits,”** Genadi Levin establishes a precise form of this relation for rational maps with one critical point satisfying the *summability condition* (certain expansion rate assumption along the critical orbit). This result brings to a natural general form many previously known special cases studied over the years by many people, including the author.

One of the most profound achievements in holomorphic dynamics in the early 1980s was *Thurston’s topological characterization of rational maps*, which gives a combinatorial criterion for a postcritically finite branched covering of the sphere to be realizable (in a certain homotopical sense) as a rational map (see [DH2]). A wealth of new powerful ideas from hyperbolic geometry and Teichmüller theory were introduced to the field in this work. The *Thurston Rigidity Theorem*, which gives uniqueness of the realization, although only a small part of the theory, already is a major insight, with many important consequences for the field (some of which are mentioned later).

Attempts to generalize Thurston’s characterization to the transcendental case faces many difficulties. However, in the exponential family $z \mapsto e^{\lambda z}$, they were overcome by Hubbard, Schleicher and Shishikura [HSS]. In the paper **“A framework towards understanding the characterization of holomorphic dynamics,”** Yunping Jiang surveys these and further results, which, in particular, extend the theory to a certain class of postcritically infinite maps. His paper includes an appendix by the author, Tao Chen, and Linda Keen that proposes applications of the ideas developed on the survey to the characterization problem for certain families of quasi-entire and quasi-meromorphic functions.



The field of *real one-dimensional dynamics* emerged from obscurity in the mid-1970s, largely due to the seminal work by Milnor and Thurston [MT], where they laid down foundations of the combinatorial theory of one-dimensional dynamics, called *kneading theory*. To any piecewise monotone interval map f , the authors associated a topological invariant (determined by the ordering of the critical orbits on the line) called the *kneading invariant*, which essentially classifies the maps in question. Another important invariant, the *topological entropy* $h(f)$ (which measures “the complexity” of a dynamical system) can be read off from the kneading invariant. One of the conjectures posed in the preprint version of [MT] was that in the real quadratic family $f_a : x \mapsto ax(1-x)$, $a \in (0, 4]$, the topological entropy depends monotonically on a . This conjecture was proved in the final version using methods of holomorphic dynamics (the Thurston Rigidity Theorem alluded to earlier). This was the first occasion that demonstrated how fruitful complex methods could be in real dynamics. Much more was to come: see, e.g., [L4], a recent survey on this subject.

Later on, Milnor posed the general *monotonicity conjecture* [M6] (compare [DGMTr]) asserting that *in the family of real polynomials of any degree, isentropes are connected* (where an *isentrope* is the set of parameters with the same entropy). This conjecture was proved in the cubic case by Milnor and Tresser [MTr], and in the general case by Bruin and van Strien [BvS]. In the survey “**Milnor’s conjecture on monotonicity of topological entropy: Results and questions,**” Sebastian van Strien discusses the history of this conjecture, gives an outline of the proof in the general case, and describes the state of the art in the subject. The proof makes use of an important result by Kozlovski, Shen, and van Strien [KSvS] on the density of hyperbolicity in the space of real polynomial maps, which is a far-reaching generalization of the Thurston Rigidity Theorem. (In the quadratic case, density of hyperbolicity had been proved in [L3, GrSw].) The article concludes with a list of open problems.

The paper “**Entropy in dimension one**” is one of the last papers written by William Thurston and occupies a special place in this volume. Sadly, Bill Thurston passed away in 2012 before finishing this work. In this paper, Thurston studies the topological entropy h of postcritically finite one-dimensional maps and, in particular, the relations between dynamics and arithmetics of e^h , presenting some amazing constructions for maps with given entropy and characterizing what values of entropy can occur for postcritically finite maps. In particular, he proves: *h is the topological entropy of a postcritically finite interval map if and only if $h = \log \lambda$, where $\lambda \geq 1$ is a weak Perron number, i.e., it is an algebraic integer, and $\lambda \geq |\lambda^\sigma|$ for every Galois conjugate $\lambda^\sigma \in \mathbb{C}$.*

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In the mid-1980s, Milnor wrote a short conceptual article “*On the concept of attractor*” [M5] that made a substantial impact on the field of real one-dimensional dynamics. In this paper Milnor proposed a general notion of *measure-theoretic attractor*, illustrated it with the *Feigenbaum attractor*, and formulated a problem of existence of *wild attractors* in dimension one. Such an attractor would be a Cantor set that attracts almost all orbits of some topologically transitive periodic