Frontiers in Complex Dynamics

In Celebration of John Milnor's 80th Birthday

EDITED BY ARACELI BONIFANT,
MISHA LYUBICH,
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Frontiers in Complex Dynamics

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Preface

In February of 2011, over 130 mathematicians gathered at the beautiful Banff Centre in the Canadian Rockies for a week of discussing holomorphic dynamics in one and several variables and other topics related to the work of John Milnor.

John Milnor is undoubtedly one of the most significant mathematicians of the second half of the twentieth century. He has made fundamental discoveries in many areas of modern mathematics, including topology, geometry, K-theory, and dynamical systems. Since in recent years his main interest has been in complex dynamics, it was only fitting that the conference had this as a primary focus.

The conference in Banff was a great success. In addition to all of the wonderful mathematics, the beautiful setting and friendly atmosphere at the Banff Centre inspired us all, both professionally and personally. All but one of the talks were videotaped and can be viewed or downloaded from the conference website, at `http://www.math.sunysb.edu/jackfest` (unfortunately, there was a camera malfunction at the start of Arnaud Chéritat's presentation, so only his slides are available).

This collection is an outgrowth of that conference, which was also organized by the editors of this book. Almost all of the authors whose papers appear here attended the conference. Both this collection and the conference were designed to honor John Milnor. But this volume is not merely a record of that conference; rather, it extends and complements that event. For example, some of the speakers gave primarily expository lectures but chose to contribute research papers to this volume; others went the other route. There is very little skiing in the book, and you'll have to bring your own food. But, it should last longer. We hope this volume will be valuable to any mathematician working in complex dynamics or related fields, whether or not they attended the Banff conference.

This volume is organized in five main parts: ***I. One Complex Variable, II. One Real Variable, III. Several Complex Variables, IV. Laminations and Foliations,*** and ***V. Geometry and Algebra***. The first part is further subdivided into the areas of *arithmetic dynamics, polynomial dynamics, rational dynamics,* and *Thurston theory,* and the third part first covers *local dynamics in several complex variables* and then turns to *global dynamics.* In addition, there is a section containing color versions of those images for which color is essential; such images have references within the body of the main text, where a greyscale version appears for the reader's convenience.

The photograph of Jack Milnor at Lake Louise (Plate 1) was taken by Thomas Milnor. The images in Figure 1.2 on page 75 (which appears in modified form as Plate 4) are reprinted from John Milnor's article "Remarks on Iterated Cubic

Maps" in *Experimental Mathematics* 1, no. 1 (1992) by permission of Taylor & Francis (http://www.tandfonline.com). The image in Plate 15 (which also appears in modified form as Figure 3.11 on page 151) was produced by Hiroyuki Inou. The images in Plate 23 and Figure 1.1 on page 465 were produced by Vincent Pit. The group photo of conference participants (Plates 29 and 30) was taken by Photographic Services, The Banff Centre. The conference poster on page C-24 used elements from photographs taken by Marco Martens and by Tom Arban Photography (http://www.tomarban.com). All images are used by permission.

The editors are grateful to editorial staff at Princeton University Press for their help in the production of this volume. We are also indebted to the Banff Centre and the Banff International Research Station (BIRS) for hosting the conference, as well as to the National Science Foundation, the Simons Foundation, BIRS, IBM, the University of Rhode Island, and Stony Brook University for providing financial and other support to make the conference possible (and consequently also this volume). And finally, we are especially indebted to Jack Milnor for decades of inspirational mathematics, friendship, and personal support. We look forward to many more birthday celebrations!

Contents

Introduction

Holomorphic dynamics is one of the earliest branches of dynamical systems which is not part of classical mechanics. As a prominent field in its own right, it was founded in the classical work of Fatou and Julia (see [Fa1, Fa2] and [J]) early in the 20th century. For some mysterious reason, it was then almost completely forgotten for 60 years. The situation changed radically in the early 1980s when the field was revived and became one of the most active and exciting branches of mathematics. John Milnor was a key figure in this revival, and his fascination with holomorphic dynamics helped to make it so prominent. Milnor's book *Dynamics in One Complex Variable* [M8], his volumes of collected papers [M10, M11], and the surveys [L1, L5] are exemplary introductions into the richness and variety of Milnor's work in dynamics.

Holomorphic dynamics, in the sense we will use the term here, studies iterates of holomorphic maps on complex manifolds. Classically, it focused on the dynamics of rational maps of the Riemann sphere $\widehat{\mathbb{C}}$. For such a map f, the Riemann sphere is decomposed into two invariant subsets, the *Fatou set* $\mathcal{F}(f)$, where the dynamics is quite tame, and the *Julia set* $\mathcal{J}(f)$, which often has a quite complicated fractal structure and supports chaotic dynamics.

Even in the case of quadratic polynomials $Q_c\colon z \mapsto z^2 + c$, the dynamical picture is extremely intricate and may depend on the parameter c in an explosive way. The corresponding bifurcation diagram in the parameter plane is called the *Mandelbrot set*; its first computer images appeared in the late 1970s, sparking an intense interest in the field [BrMa, Man].

The field of holomorphic dynamics is rich in interactions with many branches of mathematics, such as complex analysis, geometry, topology, number theory, algebraic geometry, combinatorics, and measure theory. The present book is a clear example of such interplay.

The papers **"Arithmetic of Unicritical Polynomial Maps"** and **"Les racines de composantes hyperboliques de M sont des quarts d'entiers algébriques,"** which open this volume,[1] exemplify the interaction of holomorphic dynamics with number theory. In these papers, John Milnor and Thierry Bousch study number-theoretic properties of the family of polynomials $p_c(z) = z^n + c$, whose bifurcation diagram is known as the *Multibrot set.*

In the celebrated Orsay Notes [DH1], Douady and Hubbard undertook a remarkable combinatorial investigation of the Mandelbrot set and the corresponding bifurcations of the Julia sets. In particular, they realized (using important contributions from Thurston's work [T]) that these fractal sets admit an explicit topological model as long as they are locally connected (see [D]). This led to the most famous conjecture in the field, on the local connectivity of the Mandelbrot set, typically

[1] Both these papers were originally written circa 1996 but never published. Milnor's paper is a follow-up to Bousch's note, but it was significantly revised by the author for this volume.

abbreviated as *MLC*. The MLC conjecture is still currently open, but it has led to many important advances, some of which are reflected in this volume.

In his thesis [La], Lavaurs proved the non-local-connectivity of the cubic connectedness locus, highlighting the fact that the degree two case is special in this respect. In attempt to better understand this phenomenon, Milnor came across a curious new object that he called the *tricorn*: the connectedness locus of antiholomorphic quadratic maps $q_c(z) = \bar{z}^2 + c$. In the paper **"Multicorns are not path connected,"** John Hubbard and Dierk Schleicher take a close look at the connectedness locus of its higher degree generalization, defined by $p_c(z) = \bar{z}^n + c$.

The paper by Alexandre Dezotti and Pascale Roesch, **"On (non-)local connectivity of some Julia sets,"** surveys the problem of local connectivity of Julia sets. It collects a variety of results and conjectures on the subject, both "positive" and "negative" (as Julia sets sometimes fail to be locally connected). In particular, in this paper the reader can learn about the work of Yoccoz [H, M7], Kahn and Lyubich [KL], and Kozlovski, Shen, and van Strien [KSvS]; the latter gives a positive answer in the case of "non-renormalizable" polynomials of any degree.

Related to connectivity, an important question that has interested both complex and algebraic dynamicists is that of the irreducibility of the closure of X_n, the set of points $(c, z) \in \mathbb{C}^2$ for which z is periodic under $Q_c(z) = z^2 + c$ with minimal period n. These curves are known as *dynatomic curves*. The irreducibility of such curves was proved by Morton [Mo] using algebraic methods, by Bousch [Bou] using algebraic and analytic (dynamical) methods, and by Lau and Schleicher [LS], using only dynamical methods. In the paper **"The quadratic dynatomic curves are smooth and irreducible,"** Xavier Buff and Tan Lei present a new proof of this result based on the *transversality theory* developed by Adam Epstein [E].

Similarly, in the case of the family of cubic polynomial maps with one marked critical point, parametrized by the equation $F(z) = z^3 - 3a^2z + (2a^3 + v)$, one can study the *period p-curves* $\mathcal{S}_p$ for $p \geq 1$. These curves are the collection of parameter pairs $(a, v) \in \mathbb{C}^2$ for which the marked critical point a has period exactly p; Milnor proved that $\mathcal{S}_p$ is smooth and affine for all $p > 0$ and irreducible for $p \leq 3$ [M9]. The computation of the Euler characteristic for any $p > 0$ and the irreducibility for $p = 4$ were proved by Bonifant, Kiwi and Milnor [BKM]. The computation of the Euler characteristic requires a deep study of the unbounded hyperbolic components of $\mathcal{S}_p$, known as *escape regions*. Important information about the limiting behavior of the periodic critical orbit as the parameter tends to infinity within an escape region is encoded in an associated *leading monomial vector*, which uniquely determines the escape region, as Jan Kiwi shows in **"Leading monomials of escape regions."**

As we have alluded to previously, a locally connected Julia set admits a precise topological model, due to Thurston, by means of a *geodesic lamination* in the unit disk. This model can be efficiently described in terms of the *Hubbard tree*, which is the "core" that encodes the rest of the dynamics. In particular, it captures all the cut-points of the Julia set, which generate the lamination in question. This circle of ideas is described and is carried further to a more general topological setting in the paper by Alexander Blokh, Lex Oversteegen, Ross Ptacek and Vladlen Timorin **"Dynamical cores of topological polynomials."**

The realm of general *rational dynamics* on the Riemann sphere is much less explored than that of polynomial dynamics. There is, however, a beautiful bridge connecting these two fields called *mating*: a surgery introduced by Douady and

Hubbard in the 1980s, in which the filled Julia sets of two polynomials of the same degree are dynamically related via external rays. In many cases this process produces a rational map. It is a difficult problem to decide when this surgery works and which rational maps can be obtained in this way. A recent breakthrough in this direction was achieved by Daniel Meyer, who proved that in the case when f is postcritically finite and the Julia set of f is the whole Riemann sphere, every sufficiently high iterate of the map can be realized as a mating [Me1, Me2]. In the paper **"Unmating of rational maps, sufficient criteria and examples,"** Meyer gives an overview of the current state of the art in this area of research, illustrating it with many examples. He also gives a sufficient condition for realizing rational maps as the mating of two polynomials.

Another way of producing rational maps is by "singular" perturbations of complex polynomials. In the paper **"Limiting behavior of Julia sets of singularly perturbed rational maps,"** Robert Devaney surveys dynamical properties of the families $f_{c,\lambda}(z) = z^n + c + \lambda/z^d$ for $n \geq 2$, $d \geq 1$, with c corresponding to the center of a hyperbolic component of the Multibrot set. These rational maps produce a variety of interesting Julia sets, including *Sierpinski carpets* and *Sierpinski gaskets*, as well as laminations by Jordan curves. In the current article, the author describes a curious "implosion" of the Julia sets as a polynomial $p_c = z^n + c$ is perturbed to a rational map $f_{c,\lambda}$.

There is a remarkable *phase-parameter relation* between the dynamical and parameter planes in holomorphic families of rational maps. It first appeared in the early 1980s in the context of quadratic dynamics in the Orsay notes [DH1] and has become a very fruitful philosophy ever since. In the paper **"Perturbations of weakly expanding critical orbits,"** Genadi Levin establishes a precise form of this relation for rational maps with one critical point satisfying the *summability condition* (certain expansion rate assumption along the critical orbit). This result brings to a natural general form many previously known special cases studied over the years by many people, including the author.

One of the most profound achievements in holomorphic dynamics in the early 1980s was *Thurston's topological characterization of rational maps*, which gives a combinatorial criterion for a postcritically finite branched covering of the sphere to be realizable (in a certain homotopical sense) as a rational map (see [DH2]). A wealth of new powerful ideas from hyperbolic geometry and Teichmüller theory were introduced to the field in this work. The *Thurston Rigidity Theorem*, which gives uniqueness of the realization, although only a small part of the theory, already is a major insight, with many important consequences for the field (some of which are mentioned later).

Attempts to generalize Thurston's characterization to the transcendental case faces many difficulties. However, in the exponential family $z \mapsto e^{\lambda z}$, they were overcome by Hubbard, Schleicher and Shishikura [HSS]. In the paper **"A framework towards understanding the characterization of holomorphic dynamics,"** Yunping Jiang surveys these and further results, which, in particular, extend the theory to a certain class of postcritically infinite maps. His paper includes an appendix by the author, Tao Chen, and Linda Keen that proposes applications of the ideas developed on the survey to the characterization problem for certain families of quasi-entire and quasi-meromorphic functions.

The field of *real one-dimensional dynamics* emerged from obscurity in the mid-1970s, largely due to the seminal work by Milnor and Thurston [MT], where they laid down foundations of the combinatorial theory of one-dimensional dynamics, called *kneading theory*. To any piecewise monotone interval map f, the authors associated a topological invariant (determined by the ordering of the critical orbits on the line) called the *kneading invariant*, which essentially classifies the maps in question. Another important invariant, the *topological entropy* $h(f)$ (which measures "the complexity" of a dynamical system) can be read off from the kneading invariant. One of the conjectures posed in the preprint version of [MT] was that in the real quadratic family $f_a : x \mapsto ax(1-x)$, $a \in (0,4]$, the topological entropy depends monotonically on a. This conjecture was proved in the final version using methods of holomorphic dynamics (the Thurston Rigidity Theorem alluded to earlier). This was the first occasion that demonstrated how fruitful complex methods could be in real dynamics. Much more was to come: see, e.g., [L4], a recent survey on this subject.

Later on, Milnor posed the general *monotonicity conjecture* [M6] (compare [DGMTr]) asserting that *in the family of real polynomials of any degree, isentropes are connected* (where an *isentrope* is the set of parameters with the same entropy). This conjecture was proved in the cubic case by Milnor and Tresser [MTr], and in the general case by Bruin and van Strien [BvS]. In the survey **"Milnor's conjecture on monotonicity of topological entropy: Results and questions,"** Sebastian van Strien discusses the history of this conjecture, gives an outline of the proof in the general case, and describes the state of the art in the subject. The proof makes use of an important result by Kozlovski, Shen, and van Strien [KSvS] on the density of hyperbolicity in the space of real polynomial maps, which is a far-reaching generalization of the Thurston Rigidity Theorem. (In the quadratic case, density of hyperbolicity had been proved in [L3, GrSw].) The article concludes with a list of open problems.

The paper **"Entropy in dimension one"** is one of the last papers written by William Thurston and occupies a special place in this volume. Sadly, Bill Thurston passed away in 2012 before finishing this work. In this paper, Thurston studies the topological entropy h of postcritically finite one-dimensional maps and, in particular, the relations between dynamics and arithmetics of e^h, presenting some amazing constructions for maps with given entropy and characterizing what values of entropy can occur for postcritically finite maps. In particular, he proves: *h is the topological entropy of a postcritically finite interval map if and only if $h = \log \lambda$, where $\lambda \geq 1$ is a weak Perron number, i.e., it is an algebraic integer, and $\lambda \geq |\lambda^\sigma|$ for every Galois conjugate $\lambda^\sigma \in \mathbb{C}$.*

The editors received significant help from many people in preparing this paper for publication, to whom we are very grateful. Among them are M. Bestvina, M. Handel, W. Jung, S. Koch, D. Lind, C. McMullen, L. Mosher, and Tan Lei. We are especially grateful to John Milnor, who carefully studied Bill's manuscript, adding a number of notes which clarify many of the points mentioned in the paper.

In the mid-1980s, Milnor wrote a short conceptual article *"On the concept of attractor"* [M5] that made a substantial impact on the field of real one-dimensional dynamics. In this paper Milnor proposed a general notion of *measure-theoretic attractor*, illustrated it with the *Feigenbaum attractor*, and formulated a problem of existence of *wild attractors* in dimension one. Such an attractor would be a Cantor set that attracts almost all orbits of some topologically transitive periodic